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Observation et contrôle de quelques systèmes conservatifs

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Mots clés : opérateurs de sturm-liouville, problèmes extrémaux, calculs des variations, inégalité d'ingham, contrôlabilité indirecte de systèmes linéaires, équations des ondes et de schrödinger, méthode de contrôle fictif, résolubilité algébrique.

Keywords: sturm-liouville operators, extremal problems, calculus of variations, ingham's inequality, indirect controllability of linear systems, schrödinger and wave equations, fictitious control method, algebraic solvability.

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Observation et contrôle de quelques systèmes conservatifs

Résumé

Dans cette thèse, nous nous intéressons à la contrôlabilité interne et à son coût pour une ou plusieurs équations aux dérivées partielles conservatives.
Dans la première partie, nous introduisons et détaillons deux méthodes permettant d'estimer le coût du contrôle (et par dualité, de la constante d'observabilité) de l'équation des ondes avec potentiel L^∞ en dimension un d'espace. La première utilise la propagation des ondes le long des caractéristiques en s'appuyant sur le rôle symétrique de la variable de temps et d'espace. La deuxième méthode repose sur la décomposition spectrale de l'équation des ondes et sur l'utilisation des inégalités d'Ingham. L'estimation de la constante d'observabilité se ramène alors à l'étude d'un problème d'optimisation faisant intervenir les vecteurs propres du Laplacien-Dirichlet avec potentiel. Nous fournissons ensuite des propriétés qualitatives sur le minimiseurs ainsi qu'une estimation du minimum ne dépendant que de la mesure de l'ensemble d'observation.
Dans la deuxième partie, nous étudions la contrôlabilité de certains systèmes d'équations avec un nombre de contrôles réduits, autrement dit le nombre de contrôles est plus petit que le nombre d'équations. En particulier, nous caractérisons exactement les données initiales qui peuvent être contrôlées pour des systèmes d'équations couplées de type Schrödinger et nous énonçons une condition nécessaire et suffisante de type Kalman pour des systèmes d'équations des ondes couplées. La preuve repose sur une méthode de contrôle fictif combinée à la résolution algébrique d'un système sous-déterminé et sur certains résultats de régularité.

Mots clés : opérateurs de sturm-liouville, problèmes extrémaux, calculs des variations, inégalité d'ingham, contrôlabilité indirecte de systèmes linéaires, équations des ondes et de schrödinger, méthode de contrôle fictif, résolubilité algébrique.

Abstract

In this work, we focus on the internal controllability and its cost for some linear partial differential equations.
In the first part, we introduce and describe two methods to provide precise estimates of the cost of control (and by duality, of the observability constant) for general one dimensional wave equations with potential. The first one is based on a propagation argument along the characteristics relying on the symmetrical roles of the time and space variables. The second one uses a spectral decomposition of the solution of the wave equation and Ingham's inequalities. This relates the estimation of the observability constant to the study of an optimal problem involving Dirichlet eigenfunctions of Laplacian with potential. We provide some qualitative properties of the minimizers, and also precise bounds on the minimum.
In the second part, we are concerned with the controllability of some systems of equations by a reduced number of controls (i.e. the number of controls is less that the number of equations). In particular, in the case of coupled systems of Schrödinger equations, we exactly characterize the initial conditions that can be controlled and we give a necessary and sufficient condition of Kalman type for the controllability of coupled systems of wave equations. The proof relies on the *fictitious control method* coupled with the proof of an *algebraic solvability* property for some related underdetermined system, as well as on some regularity results.

Keywords: sturm-liouville operators, extremal problems, calculus of variations, ingham's inequality, indirect controllability of linear systems, schrödinger and wave equations, fictitious control method, algebraic solvability.

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Introduction générale

Dans cette thèse, nous allons nous intéresser à la contrôlabilité d'une EDP ou d'un système d'EDP de la forme

$$\begin{cases} \frac{d}{dt}y = \mathcal{L}y + Cu, \quad t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{1}$$

où $\mathcal{L}$ est un opérateur non borné, fermé à domaine dense sur un espace de Hilbert $\mathcal{H}$ qui génère un **groupe** et C est un opérateur linéaire continu d'un espace de Hilbert $\mathcal{U}$ dans $\mathcal{H}$. L'objet de la théorie du contrôle est d'agir sur l'évolution de l'état y par l'intermédiaire du "paramètre" u appelé le contrôle. L'opérateur C va donc décrire la manière dont le contrôle agit sur le système. Ainsi, une question naturelle est :

Q1 (contrôlabilité) : Est-il possible, pour un temps $T > 0$ et pour une condition initiale y_0, de trouver un contrôle u qui amène la solution y de (1) à 0 en temps T ?

S'il existe un tel contrôle u amenant y_0 à l'état final 0, nous dirons que le système (1) est contrôlable à 0. De plus, s'il existe, il n'a aucune raison d'être unique. Puisque l'ensemble des contrôles est un convexe fermé d'un espace de Hilbert, nous allons choisir le contrôle de norme $L^2((0,T),\mathcal{U})$ minimale noté $u_{\min}$. Considérons l'application Γ_ω linéaire continue définie par

$$\Gamma_\omega : \begin{array}{ccc} \mathcal{H} & \longrightarrow & L^2((0,T),\mathcal{U}) \\ y_0 & \longmapsto & u_{\min} \end{array}.$$

Sa norme est donnée par

$$\|\Gamma_\omega\| = \sup\left\{ \frac{\|u_{\min}\|_{L^2((0,T),\mathcal{U})}}{\|y_0\|_{\mathcal{H}}} \mid y_0 \in \mathcal{H} \setminus \{0\} \right\}.$$

La quantité $\|\Gamma_\omega\|$, que nous choisissons d'appeler le **coût du contrôle**, mesure l'énergie minimale dont on a besoin pour amener la solution de la condition initiale à 0.

Q2 : Si la réponse **Q1** est positive, pouvons nous déterminer une estimation du coût du contrôle $\|\Gamma_\omega\|$?

Par un argument de dualité observabilité-contrôlabilité (voir [30], [24, Théorème 2.42]), la réponse à la question **Q1** est équivalente à l'existence d'une constante $c > 0$ vérifiant, pour tout $\varphi_0 \in \mathcal{H}$,

$$c\|\varphi_0\|_H^2 \leqslant \int_0^T \|C^*\varphi(t)\|_{\mathcal{U}}^2 \, dt, \tag{2}$$

avec φ solution du système adjoint

$$\left\{\begin{array}{ll} \frac{d}{dt}\varphi = \mathcal{L}^*\varphi & t \in [0,T], \\ \varphi(0) = \varphi_0. \end{array}\right. \tag{3}$$

Notons par $c(T,a,\omega)$ la plus grande constante vérifiant (2). D'après [24, Théorème 2.42] on a $\|\Gamma_\omega\|^2 = \frac{1}{c(T,a,\omega)}$. Ainsi, la détermination d'une borne inférieure de la constante d'observabilité $c(T,a,\omega)$ nous permet de répondre à la question **Q2**. En outre, une estimation fine de $c(T,a,\omega)$ nous permet d'obtenir une mesure quantitative du caractère bien posé du problème inverse suivant : étant donné $T > 0$ et $\omega \subset (0,L)$, pouvons nous reconstruire les données initiales φ_0 et φ_1 à partir de la connaissance de $\partial_t\varphi_{|(0,T)\times\omega}$? (voir Section 1.2.1).

Estimation fine du coût du contrôle pour l'équation des ondes 1D avec potentiel L^∞

Dans la première partie de la thèse, nous allons obtenir une estimation fine du coût du contrôle pour l'équation des ondes 1D avec potentiel L^∞. Choisissons $\mathcal{H} = H_0^1(0,L) \times L^2(0,L)$, $\mathcal{U} = L^2((0,T)\times(0,L))$, $\mathcal{L} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$ et $\mathcal{C} = \begin{pmatrix} 0 \\ \mathbb{1}_\omega \end{pmatrix}$, avec $\mathbb{1}_\omega$ la fonction indicatrice d'un sous-ensemble ω de $(0,L)$, dans (1). Dans [64], en utilisant une méthode des moments, l'auteur démontre un résultat de contrôlabilité à 0 pour l'équation des ondes dès que ω est un intervalle et T est suffisamment grand, ce qui répond à la question **Q1**. En utilisant la méthode par propagation introduite dans [19, 45, 99] et la méthode spectrale basée sur les inégalités d'Ingham (voir [53]), nous allons répondre à la question **Q2** en déterminant une estimation optimale, en un certain sens (qui sera précisé dans [Chapitre 1, Section 1.4]), de $c > 0$ vérifiant, pour tout $(\varphi_0,\varphi_1) \in H_0^1(0,L)\times L^2(0,L)$,

$$c\|(\varphi_0,\varphi_1)\|_{H_0^1(0,L)\times L^2(0,L)} \leqslant \int_0^T \int_\omega \partial_t\varphi(t,x)^2\, dxdt, \tag{4}$$

avec

$$\begin{array}{ll} \partial_{tt}\varphi(t,x) - \partial_{xx}\varphi(t,x) + a(x)\varphi(t,x) = 0 & (t,x) \in (0,T)\times(0,L), \\ \varphi(t,0) = \varphi(t,\pi) = 0 & t \in [0,T], \\ \varphi(0,x) = \varphi_0(x),\ \partial_t\varphi(0,x) = \varphi_1(x) & x \in [0,L]. \end{array} \tag{5}$$

La méthode par propagation consiste à utiliser la propriété de propagation des ondes le long des caractéristiques en jouant sur le rôle symétrique de la variable de temps et de la variable d'espace tandis que la méthode spectrale est basée sur la décomposition de la solution φ de (5) dans une base hilbertienne de vecteurs propres $(e_{a,j})_{j\in\mathbf{N}^*}$ associés à l'opérateur $-\Delta + a(\cdot)$ dans $L^2(0,L)$. Par la méthode spectrale, nous obtenons l'estimation

$$c(T,a,\omega) \geqslant C_I \inf_{j\in\mathbf{N}^*} \int_\omega e_{a,j}^2(x)dx,$$

avec C_I une constante explicite. Ainsi, la réponse à la question **Q2** résultera d'une estimation fine de la quantité

$$\inf_{0\leqslant a\leqslant M}\ \inf_{|\omega|=rL}\ \inf_{j\in\mathbf{N}^*} \int_\omega e_{a,j}(x)^2\, dx, \tag{$\mathcal{P}_{L,r,M}$}$$

avec $r \in (0,1)$. Cette estimation dépendra uniquement de la mesure de ω et sera valable pour tout ω mesurable.

Dans la deuxième partie, nous allons énoncer des résultats de contrôlabilité pour certains systèmes couplés comme, par exemple, le système d'équations de Schrödinger

$$\begin{cases} \partial_t Y = i\Delta Y + AY + \mathbb{1}_\omega BV & \text{dans} \quad (0,T)\times L^2(\Omega)^n, \\ Y(0) = Y^0, \end{cases} \tag{Syst-Sch}$$

et le système d'équations des ondes

$$\begin{cases} \partial_{tt} Y = \Delta Y + AY + \mathbb{1}_\omega BV & \text{dans} \quad (0,T)\times L^2(\Omega)^n, \\ Y(0) = Y^0, \\ \partial_t Y(0) = Y^1, \end{cases} \tag{Syst-Ondes}$$

avec $(A,B) \in \mathcal{M}_n(\mathbb{R})\times\mathcal{M}_{n,m}(\mathbb{R})$, Ω un ouvert de $\mathbb{R}^N$ et ω un sous-ouvert de Ω. Nous allons agir sur ces systèmes avec moins de contrôles que d'équations, c'est à dire en supposant que $m < n$. Nous allons donc nous intéresser à comprendre comment l'information se propage d'une équation à une autre à travers la matrice de couplage A afin de construire un contrôle V qui enverra la solution Y en 0 au temps T. Dans le cas d'un système du type (Syst-Sch), nous caractériserons exactement les conditions initiales Y^0 contrôlables alors que dans le cas d'un système du type (Syst-Ondes), nous énoncerons des conditions de type Kalman sur A et B pour que le système (Syst-Ondes) soit contrôlable en temps T.

Nous utiliserons une méthode de contrôle fictif introduite dans [38] pour prouver ce résultat. Ainsi on va décomposer la preuve en deux parties, une partie analytique et une partie algébrique. La partie analytique consistera à contrôler le système

$$\begin{cases} \partial_t Z = i\Delta Z + AZ + \mathbb{1}_\omega U \\ Z(0) = Y^0. \end{cases} \tag{6}$$

Le système (6) est plus simple à contrôler que le système (Syst-Sch) puisque nous agissons sur le système avec autant de contrôles que d'équations ($n = m$). Le contrôle U de (6) sera amené à disparaître par la suite, d'où le nom de contrôle fictif. On va ensuite enlever $n - m$ contrôles en résolvant le problème algébrique

$$\begin{cases} \partial_t X = i\Delta X + AX + B\mathbb{1}_\omega W + \mathbb{1}_\omega U \\ X(0) = X(T) = 0. \end{cases} \tag{7}$$

La résolution de (7) est basée sur une idée de [41, Section 2.3.8] qui consiste à réécrire le système (7) comme un système sous-déterminé en la variable (X,W) et de considérer $f := \mathbb{1}_\omega U$ comme un second membre. On conclut en remarquant que le couple $(Y,V) = (Z - X, -W)$ est une solution de (Syst-Sch) avec $Y(T,\cdot) = 0$.

Première partie

Quelques estimations de la constante d'observabilité pour l'équation des ondes en une dimension

Chapitre 1

Introduction

Ces résultats sont extraits de [46] (en collaboration avec Alain Haraux et Yannick Privat) et de [72] (en collaboration avec Pierre Lissy et Yannick Privat).

1.1 Mise en place des problèmes

Nous considérons l'équation des ondes en une dimension avec conditions au bord de Dirichlet

$$\begin{array}{ll} \partial_{tt}\varphi(t,x) - \partial_{xx}\varphi(t,x) + a(x)\varphi(t,x) = 0 & (t,x) \in (0,T) \times (0,L), \\ \varphi(t,0) = \varphi(t,L) = 0 & t \in [0,T], \\ \varphi(0,x) = \varphi_0(x),\ \partial_t\varphi(0,x) = \varphi_1(x) & x \in [0,L]. \end{array} \tag{1.1}$$

Nous supposons que le potentiel $a(\cdot)$ est une fonction positive appartenant à $L^\infty(0,L)$. Soit $\mathcal{L}_a : H^2(0,L) \cap H_0^1(0,L) \subset L^2(0,L) \to L^2(0,L)$ l'opérateur défini par

$$\mathcal{L}_a := -\partial_{xx} + a(\cdot).$$

Pour toutes données initiales $(\varphi_0, \varphi_1) \in H_0^1(0,L) \times L^2(0,L)$, il existe une unique solution $\varphi \in C^0(0,T; H_0^1(0,L)) \cap C^1(0,T; L^2(0,L))$ au problème de Cauchy (1.1) (voir [81, p. 222]). De plus, puisque l'opérateur $\mathcal{L}_a$ est autoadjoint positif à résolvante compacte, il admet une base hilbertienne de $L^2(0,L)$ constituée de vecteurs propres $e_{a,j}$ et il existe une suite strictement croissante de nombres réels positifs $(\lambda_j)_{j\in\mathbb{N}^*}$ telle que $e_{a,j}$ résout le problème aux valeurs propres

$$\begin{cases} -e''_{a,j}(x) + a(x)e_{a,j}(x) = \lambda_j^2 e_{a,j}(x),\ x \in (0,L), \\ \qquad e_{a,j}(0) = e_{a,j}(L) = 0. \end{cases} \tag{1.2}$$

Pour obtenir l'unicité nous choisissons de prendre $e'_{a,j}(0) > 0$ et nous imposons une condition de normalisation L^2

$$\int_0^L e_{a,j}^2(x)\, dx = 1.$$

Soit

$$E_a(\varphi_0, \varphi_1) = \int_0^L \left(\varphi_1(x)^2 + \varphi_0'(x)^2 + a(x)\varphi_0(x)^2\right)\, dx,$$

l'énergie initiale du système qui est conservée au cours du temps. Rappelons la notion d'observabilité :

Définition 1 (Inégalité d'observabilité pour l'équation des ondes)**.** *Soit ω un sous-ensemble mesurable de $(0, L)$. L'équation (1.1) est dite observable sur ω en temps T s'il existe une constante strictement positive c telle que,*

$$\int_0^T \int_\omega \partial_t \varphi(t,x)^2 \, dxdt \geqslant c E_a(\varphi_0, \varphi_1), \tag{1.3}$$

pour tout $(\varphi_0, \varphi_1) \in H_0^1(0,L) \times L^2(0,L)$.

Si (1.1) est observable sur ω en temps T, nous désignons par $c(T, a, \omega)$ la plus grande constante dans (1.3), c'est à dire

$$c(T,a,\omega) = \inf_{\substack{(\varphi_0,\varphi_1)\in H_0^1(0,L)\times L^2(0,L) \\ (\varphi_0,\varphi_1)\neq(0,0)}} \frac{\int_0^T \int_\omega \partial_t \varphi(t,x)^2 \, dxdt}{E_a(\varphi_0,\varphi_1)}. \tag{1.4}$$

Remarque 1. *D'après [43], si $\omega = (\alpha, \beta)$ est un intervalle inclus dans $(0, L)$ alors pour tout $T > 2\max(\alpha, L - \beta)$,*

$$c(T, a, \omega) > 0.$$

Nous allons chercher à déterminer une borne inférieure de la constante d'observabilité $c(T, a, \omega)$ définie en (1.4). En particulier, nous obtiendrons une minoration de la version spectrale de cette constante. Les motivations de cette étude seront fournies dans la Section 1.2.

Problèmes :

P1) Déterminer une constante **explicite** $c_1 > 0$ dépendant uniquement des paramètres du problème telle que

$$c(T, a, \omega) \geqslant c_1.$$

P2) Déterminer une constante **explicite** $c_2 > 0$ dépendant uniquement des paramètres du problème telle que

$$\inf_{0\leqslant a\leqslant M} \inf_{|\omega|=rL} \inf_{j\in\mathbb{N}^*} \int_\omega e_{a,j}(x)^2 \, dx \geqslant c_2 \tag{$\mathcal{P}_{L,r,M}$}$$

avec $M \in \mathbb{R}_+$ et $r \in [0, 1]$.

Les estimations résultant de l'étude du problème **P2** constitueront, en particulier, un ingrédient important pour la résolution du problème **P1**, i.e. la détermination d'un minorant de $c(T, a, \omega)$ (voir Section 1.3 et Section 1.5.1).

Les paramètres du problème seront le temps d'observation T, la norme infinie du potentiel a et la mesure de Lebesgue du domaine d'observation ω.

1.2 Motivations

Avant d'exposer nos résultats, nous allons nous intéresser aux motivations qui nous ont amené à résoudre les problèmes **P1** et **P2**.

1.2.1 Problème inverse et choix des paramètres physiques

Supposons que nous disposons de la mesure de $\partial_t \varphi$ sur le cylindre $(0,T)\times\omega$. Nous considérons le problème inverse suivant : étant donné $T>0$ et $\omega\subset(0,L)$, pouvons nous reconstruire les données initiales φ_0 et φ_1 à partir de la connaissance de $\partial_t\varphi_{|(0,T)\times\omega}$? La réponse est positive lorsque l'inégalité d'observabilité (1.3) est vérifiée, c'est à dire lorsque $c(T,a,\omega)>0$. En réalité, à cause des erreurs de mesures, on observe plutôt $\partial_t\varphi^{obs}$ sur $(0,T)\times\omega$ où φ^{obs} est solution de (1.1) avec conditions initiales φ_0^{obs} et φ_1^{obs}. L'inégalité d'observabilité nous donne alors

$$E_a(\varphi_0-\varphi_0^{obs},\varphi_1-\varphi_1^{obs})\leqslant\frac{\int_0^T\int_\omega\left(\partial_t\varphi(t,x)-\partial_t\varphi^{obs}(t,x)\right)^2\,dxdt}{c(T,a,\omega)}.$$

Ainsi la connaissance d'une borne inférieure explicite de la constante d'observabilité $c(T,a,\omega)$ nous permet d'avoir une estimation de l'erreur de reconstruction des données initiales. En résumé, la constante d'observabilité peut être interprétée comme une mesure quantitative du caractère bien posé du problème inverse de reconstruction de la solution à partir des mesures sur ω pendant un temps T.

De plus, lorsque nous faisons des expériences, à cause de certaines imprécisions sur les conditions de fonctionnement ou sur les mesures physiques, nous connaissons souvent des informations partielles entachées d'erreurs sur les paramètres du problème inverse. Par exemple, nous pouvons mentionner l'exemple de la tomographie thermoacoustique où l'intensité des mesures est souvent très faible et le modèle physique est souvent simplifié avant d'exploiter les mesures. C'est pourquoi il est intéressant d'obtenir des estimations de la constante d'observabilité $c(T,a,\omega)$ ne dépendant que du temps d'observation T, de la borne supérieure du potentiel a et de la mesure de Lebesgue du sous-ensemble d'observation ω.

1.2.2 Le coût du contrôle pour la méthode HUM

La constante d'observabilité $c(T,a,\omega)$ joue aussi un rôle important en théorie du contrôle. Considérons l'équation des ondes sur $L^2(0,L)$ avec conditions au bord de Dirichlet

$$\begin{cases}\partial_{tt}y(t,x)-\partial_{xx}y(t,x)+a(x)y(t,x)=u(t,x), & \textit{dans}\quad(0,T)\times(0,L),\\ y(t,0)=y(t,L)=0, & t\in[0,T],\\ (y(0,x),\partial_t y(0,x))=(y_0(x),y_1(x)), & x\in[0,L],\end{cases}\tag{1.5}$$

où $u\in L^2((0,T),(0,L))$ est un contrôle à support dans $[0,T]\times\omega$ avec ω est un sous-ensemble mesurable de $(0,L)$. Pour toutes données initiales $(\varphi_0,\varphi_1)\in H_0^1(0,L)\times L^2(0,L)$, il existe une unique solution $\varphi\in C^0([0,T];H_0^1)\cap C^1([0,T];L^2(0,L))$ au problème de Cauchy (1.5). Rappelons la définition de la contrôlabilité à zéro :

Définition 2. *L'équation* (1.5) *est* ***contrôlable à*** 0 ***en temps*** T *si et seulement si pour tout* (y_0,y_1) *dans* $H_0^1(0,L)\times L^2(0,L)$*, il existe un contrôle* u *dans* $L^2((0,T),(0,L))$ *tel que la solution de* (1.5) *correspondant à la condition initiale* $(y_0,y_1)\in H_0^1(0,L)\times L^2(0,L)$ *satisfasse* $y(T,\cdot)=0$.

Lorsque $c(T,a,\omega)>0$, la méthode HUM (voir [73, 74]) permet de montrer que l'équation (1.5) est contrôlable à 0 en temps T en construisant un contrôle $u_{\min}$ de norme $L^2((0,L)\times\omega)$

minimale. Soit Γ_ω l'opérateur HUM défini par

$$\Gamma_\omega : \begin{array}{rcl} H_0^1(0,L)\times L^2(0,L) & \longrightarrow & L^2((0,T)\times(0,L)) \\ (y_0,y_1) & \longmapsto & u_{\min} \end{array}$$

où $u_{\min}$ est le contrôle minimisant la norme $L^2((0,T)\times\omega)$. La norme de l'opérateur Γ_ω donnée par

$$\|\Gamma_\omega\| = \sup\left\{ \frac{\|u_{\min}\|_{L^2((0,T)\times(0,L))}}{\|(y_0,y_1)\|_{H_0^1(0,L)\times L^2(0,L)}} \mid (y_0,y_1)\in H_0^1(0,L)\times L^2(0,L)\setminus\{(0,0)\}\right\},$$

est appelée le **coût du contrôle**. Elle mesure l'énergie minimale dont nous avons besoin pour amener la solution de la condition initiale à $(0,0)$. Lorsque l'inégalité d'observabilité (1.3) est vérifiée, d'après [24, Théorème 2.42] on a

$$\|\Gamma_\omega\|^2 = \frac{1}{c(T,a,\omega)}. \tag{1.6}$$

Nous concluons donc que l'estimation de la constante d'observabilité nous fournit une estimation du coût du contrôle donnée par la méthode HUM.

1.2.3 Localisation des vecteurs propres en haute fréquence

Nous allons maintenant introduire la notion de non-localisation provenant de [40] : Nous considérons toujours une base hilbertienne de $L^2(\Omega)$ formée de vecteurs propres $e_{a,j}$ L^2-normalisés associés à $-\Delta + a(\cdot)$ sur Ω.

Définition 3 (Non-localisation)**.** *Soit $\omega\subset\Omega$ un sous-ensemble mesurable. On dit que la famille $(e_{a,j})_{j\in\mathbf{N}^*}$ n'est pas localisée si*

$$\inf_{j\in\mathbf{N}^*}\int_\omega e_{a,j}^2(x)\,dx > 0, \quad \textit{pour tout } \omega\subset\Omega \textit{ de mesure positive.}$$

Ainsi, une famille est dite non-localisée si ses éléments se "répartissent" sur tout le domaine. Les articles [42, 82, 85] nous fournissent un exemple de non-localisation en une dimension avec potentiel nul.

Exemple 1 (Non-localisation, $a=0$)**.** *On a*

$$\inf_{|\omega|=\varepsilon\pi}\inf_{j\in\mathbf{N}^*}\frac{2}{\pi}\int_\omega \sin(jx)^2\,dx = \inf_{j\in\mathbf{N}^*}\frac{2}{\pi}\int_{\omega^*}\sin(jx)^2\,dx = \frac{\varepsilon\pi-\sin(\varepsilon\pi)}{\pi} > 0,$$

pour tout $\varepsilon\in(0,1)$ et $\omega^*=\{\sin(jx)^2\leqslant\tau\}$ à mesure de Lebesgue nulle près où τ est choisie telle que $|\{\sin(jx)^2\leqslant\tau\}|=\varepsilon\pi$ (voir la Figure 1.1).

La résolution du problème **P2** permettra d'obtenir une mesure quantitative de la non-localisation des vecteurs propres de l'opérateur $\mathcal{L}_a = -\partial_{xx}+a(\cdot)$ avec conditions au bord de Dirichlet. A l'inverse, l'article [40, Section 7.7.1] exhibe un contre-exemple en considérant les vecteurs propres $(e_{0,j,k})_{(j,k)\in(\mathbf{N}^*)^2}$ sur un disque.

Contre-exemple 1 (Localisation dans un disque, $a=0$)**.** *Soit Ω le disque unité de $\mathbb{R}^2$. Fixons $k\in\mathbb{N}^*$, il existe une suite de sous-ensembles $\omega_{n,k}$ tendant vers Ω quand n tend vers l'infini telle*

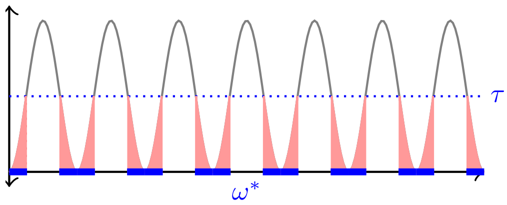


FIGURE 1.1 – $j = 7$ et $\tau = 0.5$. Répartition de ω^* et $\int_{\omega^*} \sin(jx)^2\, dx$ sur $(0, \pi)$.

que, pour n assez grand,

$$\|e_{0,n,k}\|_{L^2(\omega_{n,k})} \leqslant Cn^{\frac{2}{3}} \exp(-n^{\frac{1}{3}} \ln(2)/3).$$

Ce qui implique que pour tout sous ouvert ω inclus de facon compacte dans Ω

$$\inf_{n,k} \int_\omega e^2_{0,n,k}(x)\, dx = 0.$$

Ainsi, quand n augmente une sous-suite de vecteurs propres se concentre sur le bord (voir la Figure 1.2).

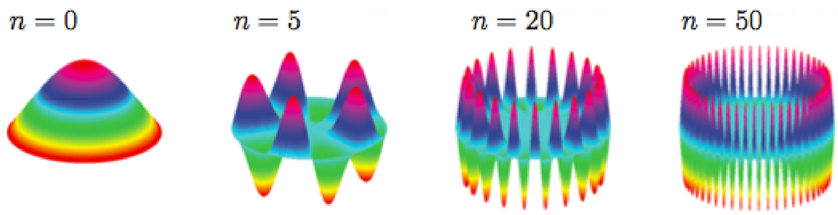


FIGURE 1.2 – Répartition de $e_{0,n,1}$ sur le disque par rapport à n.

Nous allons exposer les différentes méthodes connues permettant de résoudre le problème **P1**.

1.3 Résultats antérieurs en une dimension

Plusieurs méthodes (semi-)constructives permettent de prouver l'inégalité d'observabilité (1.3). Une première approche proposée par A. Haraux et E. Zuazua dans [19, 45, 99] consiste à utiliser la propriété de propagation des ondes le long des caractéristiques en jouant sur le rôle symétrique de la variable de temps et de la variable d'espace. En supposant que $L = \pi$ et $\omega = (\alpha, \beta)$, les auteurs obtiennent, pour tout $T > 2\max(\alpha, \pi - \beta)$

$$c(T, a, \omega) \geqslant C \exp(-\|a\|_\infty^{\frac{1}{2}}), \tag{1.7}$$

avec $C > 0$ une constante explicitable [1] ne dépendant pas de $\|a\|$. Cette estimation a permis de montrer un résultat de contrôlabilité pour des équations des ondes semi-linéaires dans [99]. Nous étendons ce résulat pour ω une union d'intervalles et en optimisant la valeur des constantes pour chaque étape de la démonstration de [45] (voir par exemple [Chapitre 2, Remarque 11]) nous obtenons une constante explicite C dans (1.7). Nous pouvons aussi utiliser la méthode des multiplicateurs introduite par L. F. Ho dans [51] pour obtenir une borne explicite de $c(T, a, \omega)$. Cependant cette méthode est trop restrictive puisqu'elle impose que l'ouvert d'observation ω touche le bord et que le temps d'observation T soit grand. Nous renvoyons à [60, 61] pour plus de détails.

Une autre approche consiste à utiliser la décomposition spectrale de l'équation des ondes. Elle donne lieu à deux méthodes :

— La méthode des moments estime le coût du contrôle en construisant une suite $\{\sigma_k(t), \tilde{\sigma}_k(t)\}$ dans $L^2(0,T)$ biorthogonale à $\{\sin(\sqrt{\lambda_k}t), \cos(\sqrt{\lambda_k}t)\}$. Les articles [64, 94] démontrent que, pour tout sous-intervalle ω de $(0, L)$ et pour T suffisamment grand,

$$c(T, a, \omega) \geqslant C_M \inf_{j\in\mathbb{N}^*} \int_\omega e_{a,j}^2(x)dx, \tag{1.8}$$

avec $C_M > 0$ une constante non explicite puisqu'elle dépend de la suite $\{\sigma_k(t), \tilde{\sigma}_k(t)\}$.

— Les inégalités d'Ingham [53] démontrent directement que, pour tout sous-ensemble ω mesurable de $(0, L)$ et pour T suffisamment grand,

$$c(T, a, \omega) \geqslant C_I \inf_{j\in\mathbb{N}^*} \int_\omega e_{a,j}^2(x)dx, \tag{1.9}$$

avec $C_I > 0$ une constante explicite donnée dans [53]. Notre théorème spectral sera basé sur cette méthode.

Ces deux méthodes spectrales nous ramènent donc à l'étude du problème

$$\inf_{j\in\mathbb{N}^*} \int_\omega e_{a,j}^2(x)dx. \tag{1.10}$$

Nous renvoyons à la Section 1.2.3 pour l'estimation de (1.10) dans le cas $a = 0$.

Remarque 2 (en dimension supérieure)**.** *Soit Ω un ouvert de $\mathbb{R}^N$ avec $N \geqslant 2$.*

— *Supposons que (ω, T) satisfasse GCC [2] (qui est une condition (quasi-)nécessaire et suffisante de contrôlabilité) alors l'article [65] montre que*

$$c(T, a, \omega) \geqslant C \exp(-\|a\|_\infty),$$

avec $C > 0$ une constante non explicite à cause de l'utilisation d'un argument de compacité-unicité.

— *Supposons que ω touche une ''bonne'' [3] partie du bord de Ω et T est suffisamment grand*

1. Nous dirons qu'une constante est explicitable si, en reprenant chaque étape de la démonstration d'une inégalité du type (1.7), nous pouvons en exhiber une constante explicite.

2. (ω, T) satisfait la condition de contrôle géométrique (GCC) dans Ω si chaque rayon de l'optique géométrique qui se propage dans Ω et qui se reflète sur sa frontière entre dans ω en un temps plus petit que T.

3. Soit ν la normale unitaire extérieure de $\partial\Omega$. Pour tout $\epsilon > 0$ on considère O_ϵ, un voisinage ouvert d'ordre ϵ dans $\mathbb{R}^n$ de $\Gamma_0 := \{x \in \partial\Omega; (x - x_0).\nu(x) > 0\}$ et on prend

$$\omega = \omega \cap O_\epsilon.$$

alors l'article [32], en utilisant des inégalités de Carleman, montre que

$$c(T,a,\omega) \geqslant C\exp(-\|a\|_\infty^{\frac{2}{3}}),$$

avec $C>0$ une constante explicitable.

1.4 Méthode spectrale versus Méthode par propagation

Énonçons directement les théorèmes principaux (simplifiés) de cette première partie que nous commenterons par la suite :

Théorème 1 (méthode spectrale)**.** *Soit ω un sous-ensemble mesurable de $(0,L)$. Nous supposons que $\|a\|_\infty \leqslant \pi^2/L^2$, alors il existe un $T_{spec}>0$ tel que pour tout $T>T_{spec}$, (1.1) est observable sur ω avec*

$$c(T,a,\omega) \geqslant |\omega|^3 C^0_{spec}(T,\|a\|_\infty), \tag{1.11}$$

où $C^0_{spec}>0$ est une constante donnée explicitement dans [Chapitre 3, Corollaire 3] et dépendant de T et de $\|a\|_\infty$.

Théorème 2 (méthode par propagation)**.** *Soit $\omega = \bigcup_{i=1}^n(\alpha_i,\beta_i)$ avec $0<\alpha_1<\beta_1<\cdots<\alpha_n<\beta_n<L$ et $T_{prop} := 2\max_{1\leqslant i\leqslant n}\{\alpha_i-\beta_{i-1},\alpha_{i+1}-\beta_i\}$, avec la convention que $\beta_0=0$ et $\alpha_{n+1}=L$. Alors, (1.1) est observable sur ω pour tout $T>T_{prop}$ avec*

$$c(T,a,\omega) \geqslant C_{prop}(T,\omega,\|a\|_\infty),$$

où C_{prop} est une constante strictement positive donnée explicitement dans [Chapitre 2, Corollaire 1] et dépendant uniquement de T, ω et $\|a\|_\infty$.

Nous énonçons une version plus détaillée du Théorème 1 dans le [Chapitre 3, Corollaire 3] et du Théorème 2 dans le [Chapitre 2, Corollaire 1]. Le résultat spectral du [Chapitre 3, Corollaire 3] améliore de façon notable celui du [Chapitre 2, Théorème 1], par exemple au lieu de supposer que le potentiel est très proche de 0 ($\|a\|_\infty \leqslant 2.08.10^{-2}$) nous imposons uniquement que $\|a\|_\infty \leqslant \pi^2/L^2$. Cette condition nous permet d'utiliser la concavité de $e_{a,j}$, une propriété fondamentale pour la mise en place de notre méthode de preuve. À cause de l'utilisation de l'inégalité d'Ingham [53], nous choisissons un temps T suffisamment grand mais nous supposons que ω est un ensemble mesurable. De plus, nous avons l'estimation

$$\inf_{\|a\|_\infty\leqslant\pi^2/L^2}\ \inf_{|\omega|=rL} c(T,a,\omega) \leqslant \frac{T}{2}\left(\frac{|\omega|}{L} - \frac{\sin(|\omega|\frac{\pi}{L})}{\pi}\right).$$

Puisque $\frac{|\omega|}{L} - \frac{\sin(|\omega|\frac{\pi}{L})}{\pi} \sim \frac{\pi^2}{6L^3}|\omega|^3$ lorsque $|\omega|$ tend vers 0, la puissance de $|\omega|$ dans le terme de droite de l'inégalité (1.11) ne peut pas être améliorée. Ainsi, lorsque la mesure de ω est petite (ce qui est le cas le plus intéressant, au moins du point de vue des applications), la minoration de $c(T,a,\omega)$ dans (1.11) est optimale par rapport à la mesure de ω. L'estimation C_{prop} du Théorème 2 est valable pour tout potentiel a mais nécessite que ω soit une union d'intervalles. De plus, dans le cas où $\omega=(\alpha,\beta)$ alors $T_{prop} = \inf\{T>0, (\omega,T)\,\text{satisfait GCC}\}$ qui est le temps minimal d'observation. Plaçons nous dans le cas $a=0$, $T=4\pi$ et $\omega=(0,\delta)\cup(\pi-\delta,\pi)$. La Figure 1.3 donne une illustration numérique de $c(T,a,\omega)$, $|\omega|^3C^0_{spec}$ et C_{prop} par rapport à la mesure de ω. Nous constatons, par exemple, que l'estimation par la méthode par propagation décroit lorsque la mesure de ω est grande, ce qui montre que C_{prop} est une mauvaise estimation de $c(T,a,\omega)$.

La Figure 1.3 met aussi en évidence l'efficacité de l'estimation par la méthode spectrale puisque $(2\delta)^3\, C^0_{spec}$ est très proche de $c(T,a,\omega)$ lorsque la mesure de ω est petite.

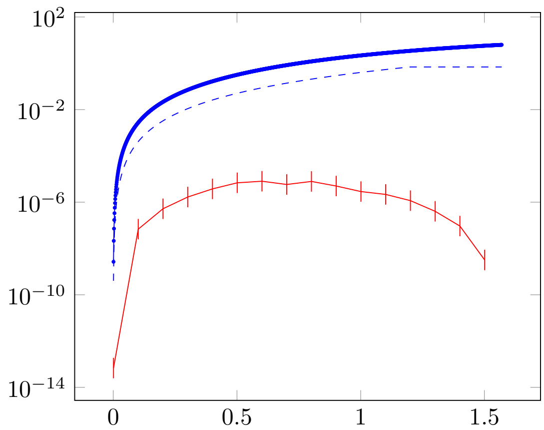


FIGURE 1.3 – $a(.) = 0$, $\omega = (0,\delta) \cup (\pi - \delta, \pi)$ et $T = 4\pi$. Courbes de $(2\delta)^3\, C^0_{spec}$ (- -), C_{prop} (//) et $c(T,a,\omega)(-)$ par rapport à δ.

Nous résumons donc les avantages et les inconvénients de chaque méthode dans le tableau suivant :

	requiert la connaissance de	Avantages	Inconvénients
Méthode spectrale	T, $\lvert\omega\rvert$, borne L^∞ sur $a(\cdot)$	- bonne estimation, - ω est seulement supposé mesurable	fonctionne pour des potentiels petits
Méthode par propagation	T, ω, borne L^∞ sur $a(\cdot)$	aucune restriction sur le potentiel	- estimation mauvaise - ω s'écrit comme une union d'intervalles

1.5 Éléments de preuves et difficultés rencontrées

1.5.1 Mise en place du problème d'optimisation P2

D'après la Section 1.3, l'étude d'un minorant de $c(T,a,\omega)$ par la méthode spectrale se ramène à la résolution du problème

$$\inf_{j\in\mathbb{N}^*} \int_\omega e_{a,j}(x)^2\, dx. \tag{1.12}$$

Le chapitre 2 est consacré à l'étude directe du problème (1.12). Fixons $j \in \mathbb{N}^*$. En utilisant une méthode de tir et le principe du minimax de Courant-Fisher nous prouvons que

$$\int_\omega e_{a,j}(x)^2\, dx \geqslant c_j,$$

avec c_j une constante définie au [Chapitre 2, (2.23)]. Pour que $\inf_{j\in\mathbb{N}^*} c_j$ soit strictement positif on a besoin d'imposer $\|a\|_\infty < 2.08.10^{-2}$, qui est une condition trop restrictive. Nous allons donc adopter dans le Chapitre 3 une approche différente en introduisant le problème d'optimisation

$$\inf_{a\in V_a} \inf_{\omega\in V_\omega} \inf_{j\in\mathbb{N}^*} \int_\omega e_{a,j}^2(x)\,dx \tag{$\mathcal{P}_{L,r,M}$}$$

avec V_ω et V_a des classes assez petites pour avoir un problème de minimisation non trivial et assez grandes pour obtenir une valeur explicite (au moins numériquement) de l'optimum. On se fixe $r \in [0,1]$ et $M \in \mathbb{R}_+$. Nous choisissons

$$V_\omega = \{\text{Sous-ensemble Lebesgue mesurable } \omega \text{ de } (0,L) \text{ tel que } |\omega| = rL\},$$

$$V_a = \{a \in L^\infty(0,L) \text{ tel que } 0 \leqslant a \leqslant M \text{ p.p. sur } (0,L)\}.$$

Si nous n'imposons pas une borne supérieure sur le potentiel a, le problème ($\mathcal{P}_{L,r,M}$) n'admet pas de solutions (voir [Chapitre 3, Théorème 4]). De plus, la contrainte sur la mesure de ω nous permet d'obtenir un problème de minimation non trivial. Ainsi les classes V_ω et V_a sont naturelles (aussi d'un point de vue physique).

1.5.2 Résolution du problème d'optimisation P2 et difficultés rencontrées

Commençons par se fixer $j \in \mathbb{N}^*$ et étudions le problème d'optimisation

$$m_j(L,r) := \inf_{0\leqslant a\leqslant M} \inf_{|\omega|=rL} \int_\omega e_{a,j}^2(x)\,dx \tag{1.13}$$

Un problème d'optimisation plus couramment étudié dans la littérature est celui de la minimation des valeurs propres du Laplacien (voir par exemple [49]). Le principe du minmax de Courant-Ficher nous permet de réécrire ce type de problème comme la minimisation d'une énergie, ce qui permet de passer outre le problème aux valeurs propres (1.2). Considérons, par exemple, le problème

$$\inf_{\Omega \text{ ouvert de } \mathbb{R}^N} \lambda_1(\Omega) = \inf_{\Omega \text{ ouvert de } \mathbb{R}^N} \min_{u\in H_0^1, u\neq 0} \frac{\int_\Omega |\nabla u(x)|^2 + a(x)|u(x)|^2\,dx}{\int_\Omega |u(x)|^2\,dx},$$

avec $\lambda_1(\Omega)$ la première valeur propre de

$$\begin{cases} -\Delta e_{a,j}(x) + a(x)e_{a,j}(x) = \lambda_j(\Omega)e_{a,j}(x), & x\in\Omega,\\ e_{a,j}(x) = 0, & x\in\partial\Omega. \end{cases}$$

Cette nouvelle formulation nous permet souvent d'obtenir des propriétés intéressantes sur le minimiseur en utilisant des arguments de réarrangements décroissants (voir [90, Chapitre 1] ,[59], [50], [49]) ou à l'aide d'un bon choix de fonctions tests. Cependant, le Problème **P2** ne peut pas se réécrire sous cette forme, ce qui est une difficulté supplémentaire. Nous écrivons donc les conditions du premier ordre associées au problème (1.13) ce qui nous amème à introduire un problème adjoint [Chapitre 3, (3.13)]. L'étude des zéros de la solution p du problème adjoint, qui permet de caractériser le potentiel optimal a du problème (1.13), est une tâche difficile puisque p est solution d'un problème de Fredholm. En utilisant la concavité de $e_{a,j}$, par une étude minitueuse du comportement variationnel de p, nous montrons que p a au moins j et au plus $3j-1$ zéros.

Ce résultat nous permet de caractériser l'optimum a^* et ω^* du problème (1.13) :
- a^* est égal à 0 ou à M presque partout sur $(0,L)$ et admet au moins j et au plus $3j-1$ points de discontinuité. (voir Figure 1.4)
- $\omega^* = \{e^2_{a^*,j} < \tau\}$ à un ensemble de mesure nulle près où τ est choisie telle que $|\{e^2_{a^*,j} < \tau\}| = rL$.

Nous réduisons donc la résolution du problème de dimension infinie (1.13) en un problème de dimension $3j-1$. En effet, nous devons minimiser la fonction :

$$(0,L)^{3j-1} \ni o = (o_1, \cdots, o_{3j-1}) \longmapsto \int_{\omega^*} e_{a^*,j}(x)^2\, dx,$$

où $(o_1, \cdots, o_{3j-1})$ sont les points de discontinuité de a^* (voir Figure 1.4). Nous résolvons numé-

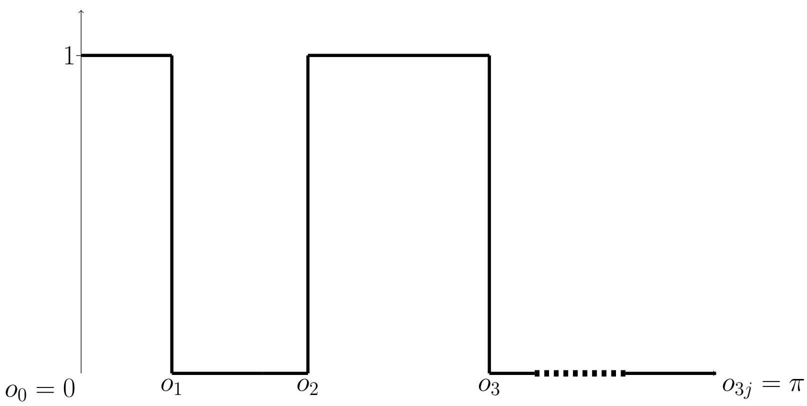


FIGURE 1.4 – Construction de $a^*(\cdot)$ avec $L = \pi$.

riquement ce problème d'optimisation par une méthode de Nelder-Mead. Nous traçons ainsi sur la Figure 1.5 les graphes de $m_j(\pi, r)$ pour $j = 1$, $j = 2$, $j = 6$ par rapport à la fraction r de la mesure de ω. Malheureusement, nous n'arrivons pas théoriquement à déterminer la valeur (ou un encadrement) de j_0 définie par

$$\inf_{j \in \mathbb{N}^*} m_j(L,r) = m_{j_0}(L,r).$$

Ainsi, pour résoudre le problème **P2** numériquement, nous devons résoudre une infinité de problèmes en dimension finie.

Fixons toujours $j \in N^*$. Nous allons donc mettre en place une nouvelle méthode pour obtenir une borne inférieure explicite du Problème (1.13). Soit $0 = x_j^0 < x_j^1 < x_j^2 < \cdots < x_j^{j-1} < L = x_j^j$ le $j+1$ zéros du j-ième vecteur propre $e_{a,j}$. Nous construisons une fonction affine par morceaux Δ_j (voir la Figure 1.6) telle que, pour tout $a \in L^\infty(0,L)$,

$$e_{a,j}(x) \geqslant \Delta_j(x) \,\text{pour tout}\, x \in (0,L) \quad \text{et} \quad \inf_{|\omega|=rL} \int_\omega \Delta_j(x)^2\, dx \geqslant \underline{m}_j,$$

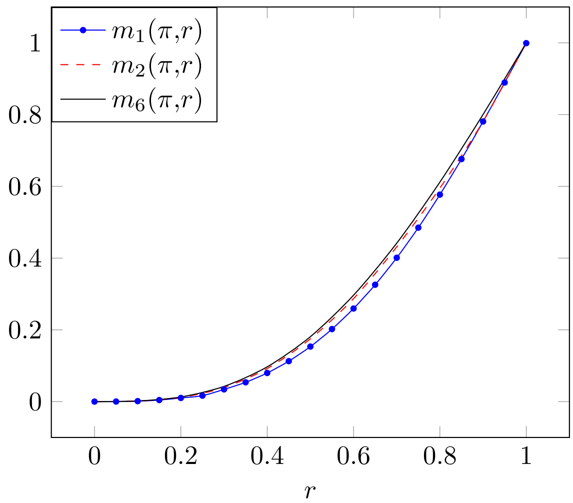


FIGURE 1.5 – $L = \pi$ et $M = 1$. Graphes de $m_j(\pi,r)$ pour $j = 1$(o), $j = 2$(- -) et $j = 6$ par rapport à r.

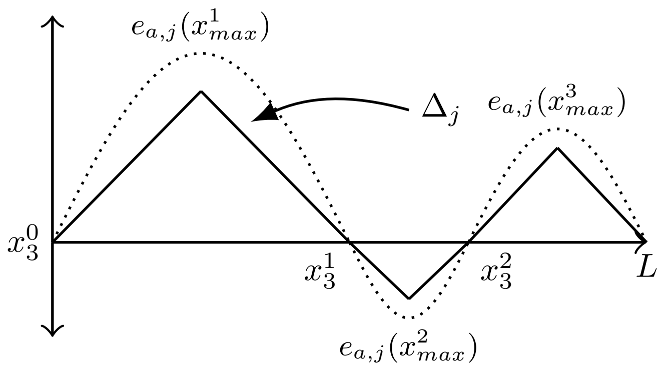


FIGURE 1.6 – Graphes de la fonction $e_{a,j}$ et $\triangle_j$ avec $j = 3$.

où $\underline{m}_j$ est une constante explicite qui sera notre candidat pour la minoration du problème (1.13). La construction Δ_j nécessite la connaissance d'une "bonne" minoration du maximum du vecteur propre $e_{a,j}$ sur chaque domaine nodal (x_j^{i-1}, x_j^i), ce qui est le point difficile de cette preuve. Le terme "bonne" signifie que la constante $\underline{m}_j$ est strictement positive pour tout $j \in \mathbb{N}^*$. Lorsque la fonction Δ_j est connue, un calcul technique nous permet de déterminer explicitement $\underline{m}_j$ (voir [Chapitre 3, Proposition 4]). On montre finalement que $j \mapsto \underline{m}_j$ est croissante pour tout $j \geqslant 2$, ce qui nous permet de résoudre le problème d'optimisation

$$\inf_{0\leqslant a\leqslant M} \inf_{|\omega|=rL} \inf_{j\in\mathbb{N}^*} \int_\omega e_{a,j}(x)^2\, dx.$$

1.6 Perspectives

Énonçons quelques extensions naturelles de notre travail :

i En reprenant nos calculs, nous pourrions chercher un entier n vérifiant

$$c(T, a, \omega) \geqslant C \exp(-\|a\|_\infty^n), \tag{1.14}$$

avec C une constante explicite et $\|a\|_\infty \leqslant \frac{\pi^2}{L^2}$. D'après [45, 99], nous devrions nous attendre à obtenir $n = \frac{1}{2}$. Puisque (1.14) est intéressant lorsque $\|a\|_\infty$ tend vers l'infini, nous pourrions essayer de supprimer la condition de bornitude sur $\|a\|_\infty$, ce qui semble une tâche difficile puisque notre méthode est basée sur la concavité de $e_{a,j}$. Une première approche consiste à supposer que $\|a\|_\infty \leqslant \lambda_{a,2}$ avec $\lambda_{a,2}$ la deuxième valeur propre du Laplacien-Dirichlet. Dans ce cas, $e_{a,j}$ est concave pour $j \geqslant 2$ et nous pouvons ainsi appliquer notre méthode "des triangles" (voir Section 1.5.2). Nous aurons donc uniquement à gérer $e_{a,1}$, qui reste un point difficile.

ii Considérons le problème **P2**

$$\inf_{j\in\mathbb{N}^*} m_j(L, r),$$

avec $m_j(L, r) := \inf_{0\leqslant a\leqslant M} \inf_{|\omega|=rL} \int_\omega e_{a,j}(x)^2\, dx$. Dans [Chapitre 3, Théorème 3], nous montrons qu'il existe un $j_0 \in \mathbb{N}^*$ vérifiant

$$\inf_{j\in\mathbb{N}^*} m_j(L, r) = m_{j_0}(L, r)$$

Numériquement, il serait intéressant de déterminer un encadrement de j_0, ce qui nous permettrait de résoudre un nombre fini de problèmes en dimension finie. De plus, la Figure 1.5 nous amène à penser que $j_0 = 1$. Dans ce cas, nous connaissons numériquement l'optimum a^* et ω^* (voir Figure 1.7). Le potentiel optimal a^* semble symétrique par rapport à $\frac{\pi}{2}$, une idée intéressante serait de construire un réarrangement de a noté a^s (voir [90, Chapitre 1] ,[59], [50], [49]) vérifiant

$$\begin{cases} -\Delta e_{a^s,j}(x) + a^s(x) e_{a^s,j}(x) = \lambda_j e_{a^s,j}(x), \ x \in \Omega, \\ e_{a^s,j}(x) = 0, \ x \in \partial\Omega, \end{cases}$$

et $\int_\omega e_{a,j}(x)^2\, dx \geqslant \int_\omega e_{a^s,j}(x)^2\, dx$, pour tout $\omega \subset (0, L)$ et pour tout $j \in \mathbb{N}^*$.

iii Une autre question intéressante serait d'obtenir une minoration de $c(T, a, \omega)$ en utilisant la méthode des moments. En s'inspirant des travaux de [97] pour construire une suite $\{\sigma_k(t), \tilde{\sigma}_k(t)\}$ dans $L^2(0, T)$ biorthogonale à $\{\sin(\sqrt{\lambda_k} t), \cos(\sqrt{\lambda_k} t)\}$, nous pourrions déterminer une estimation de la constante C_M dans (1.8). On remarque que la résolution du problème **P2** permet de considérer des sous-ensembles ω mesurable de $(0, L)$ dans (1.8).

iv Considérons le système des ondes

$$\begin{cases} \partial_{tt} y_1 = \Delta y_1 + a(x) y_1 + b(x) y_2 + \mathbb{1}_\omega u, \quad \text{sur } (0, T) \times \Omega, \\ \partial_{tt} y_2 = \Delta y_2 + b(x) y_1 + a(x) y_2 \quad \text{sur } (0, T) \times \Omega, \\ y_1(0) = y_1^0, y_2(0) = y_2^0 \quad \text{sur } (0, T) \times \partial\Omega, \end{cases} \tag{1.15}$$

D'après [3, 4], si $\{b \neq 0\}$ et ω vérifient GCC, le système (1.15) est contrôlable à 0 pour un temps T suffisamment grand. Pouvons nous déterminer une estimation fine du coût du contrôle pour (1.15) en utilisant, par exemple, une décomposition spectrale de la solution $Y = (y_1, y_2)$?

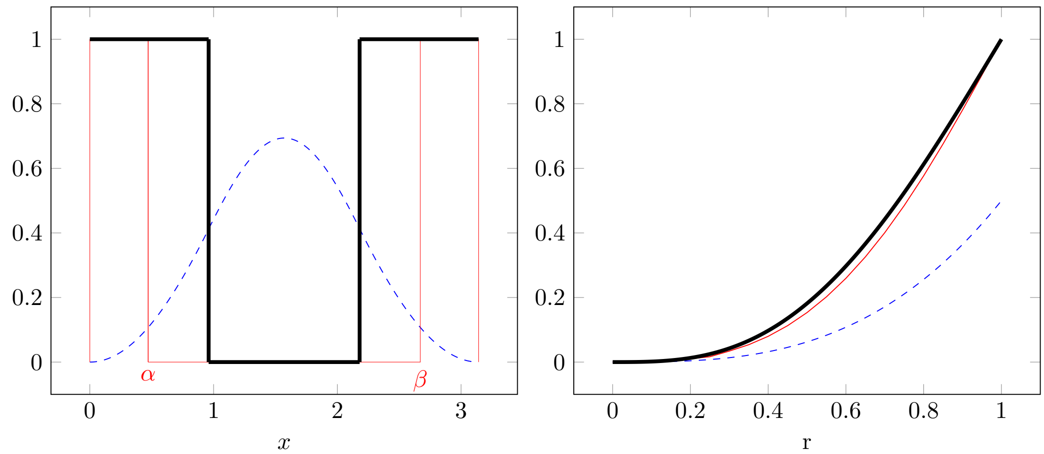


FIGURE 1.7 – $L = \pi$ et $M = 1$. Gauche : Graphes des ensembles optimaux $\omega^*(-)$, $a^*(\text{-})$ et $e^2_{a^*,1}$ $(\ldots)$ par rapport à la variable d'espace avec $r = 0.3$. Droite : Graphes de $r \mapsto m_1(L,r)$ $(-)$, $r \mapsto r - \frac{\sin(\pi r)}{\pi}$ (-) et $r \mapsto r^3/2\,(\cdots)$.

Énonçons quelques problèmes ouverts en dimension supérieure

i Il pourrait être intéressant d'étendre les résultats du problème **P2** à des ouverts $\Omega \subset \mathbb{R}^N$ avec $N \geqslant 2$. Supposons que Ω est un polygone convexe, le problème

$$\inf_{j \in \mathbf{N}^*} \int_\omega e^2_{0,j}\, dx > 0, \tag{1.16}$$

mentionné dans [40, Section 7.7.4] est ouvert. Par exemple, si Ω est un rectangle unité alors certaines valeurs propres sont dégénerées et leurs fonctions propres associées sont des combinaisons linéaires de produits de sinus et cosinus, ce qui rend la résolution du problème 1.16 difficile.

ii Considérons toujours un ouvert $\Omega \subset \mathbb{R}^N$ avec $N \geqslant 2$. Il serait intéressant de minorer la constante d'observabilité $c(T, 0, \omega)$ avec ω un sous-ouvert de Ω. D'après [52], nous nous attendons à obtenir une estimation dépendant de deux quantités :

— une quantité de nature spectrale : $J(a, \omega) := \inf_{j \in \mathbf{N}^*} \frac{\int_\omega e^2_{0,j}(x)\, dx}{\int_\Omega e^2_{0,j}(x)\, dx}$

— une quantité de nature géométrique $g_2(\omega)$ qui correspond au temps moyen passé dans ω par une géodésique pour la métrique Riemannienne, se propageant dans Ω à vitesse 1. Pour une définition précise, nous renvoyons à [52, Définition 4]. Par exemple, lorsque Ω est un rectangle unité, nous pouvons calculer explicitement $g_2(\omega)$ par un algorithme énoncé dans [48].

Chapitre 2

How to estimate observability constants of one-dimensional wave equations ? Propagation versus Spectral methods

Joint work with **Alain Haraux** and **Yannick Privat**.

Abstract : For a given bounded connected domain in $\mathbb{R}^n$, the issue of computing the observability constant associated to a wave operator, an observation time T and a generic observation subdomain constitutes in general a hard task, even for one-dimensional problems. In this work, we introduce and describe two methods to provide precise (and even sharp in some cases) estimates of observability constants for general one dimensional wave equations : the first one uses a spectral decomposition of the solution of the wave equation whereas the second one is based on a propagation argument along the characteristics. Both methods are extensively described and we then comment on the advantages and drawbacks of each one. The discussion is illustrated by several examples and numerical simulations. As a byproduct, we deduce from the main results estimates of the cost of control (resp. the decay rate of the energy) for several controlled (resp. damped) wave equations.

Keywords : wave equation, characteristics method, Sturm-Liouville problems, eigenvalues, Ingham's inequality.

2.1 Introduction and main results

2.1.1 Motivations and framework

In control theory, well-posedness issues often come down to showing that a given observability constant is positive. Nevertheless, from a practical point of view, if the considered observability constant is close to zero, the cost of control may be huge, leading to numerical instability phenomena for instance. For this reason, a precise estimate of the observability constant brings in many cases an interesting and tractable information.

In the context of inverse problems, the observability constant can be interpreted as a quantitative measure of the well-posed character of the problem. In control theory, it is directly related to the cost of control.

When realizing experiments, it may also arise, due to several imprecisions on the operating conditions or on the measures, that one only knows partial informations on the parameters of the related inverse/control problem. For instance, one could mention the example of Thermoacoustic tomography where the intensity of measures is often very weak and the physical model is in general simplified before exploiting the measures. In that context, it is interesting to obtain estimates of the observability constant depending only on some parameters of the experiment. In what follows, we concentrate on the observation of the wave equation with a zero-order potential function. As physical parameters, we choose the observability time T, possibly the Lebesgue measure of the observation subset and some L^∞ bounds on the potential function. In what follows, we will then derive estimates of the observability constant depending only on such parameters.

More precisely, this work is devoted to providing explicit lower bounds of observability constants for one-dimensional wave equations with potential using two different methods and comparing the results. We choose in the sequel potentials that only depend on the space variable since it is often more relevant in view of physical applications. A typical case is the consideration of geophysics waves influenced by the earth rotation. Let us make the frame of our study more precise.

Let T denote a positive constant standing for the observability time. We consider the one dimensional wave equation with potential

$$\begin{array}{ll} \partial_{tt}\varphi(t,x) - \partial_{xx}\varphi(t,x) + a(x)\varphi(t,x) = 0 & (t,x) \in (0,T)\times(0,\pi), \\ \varphi(t,0) = \varphi(t,\pi) = 0 & t \in [0,T], \\ \varphi(0,x) = \varphi_0(x),\ \partial_t\varphi(0,x) = \varphi_1(x) & x \in [0,\pi], \end{array} \tag{2.1}$$

where the potential $a(\cdot)$ is a nonnegative function belonging to $L^\infty(0,\pi)$. It is well known that for every initial data $(\varphi_0,\varphi_1) \in H_0^1(0,\pi)\times L^2(0,\pi)$, there exists a unique solution $\varphi \in C^0(0,T;H_0^1(0,\pi)) \cap C^1(0,T;L^2(0,\pi))$ of the Cauchy problem (2.1).

Notice that, defining the energy function E_a by

$$\begin{array}{rccl} E_a : & [0,T] & \longrightarrow & \mathbb{R}_+ \\ & t & \longmapsto & \int_0^\pi \partial_t\varphi(t,x)^2 + \partial_x\varphi(t,x)^2 + a(x)\varphi(t,x)^2\,dx, \end{array}$$

there holds

$$E_a(0) = E_a(t) \tag{2.2}$$

for every $t \in [0,T]$ and every solution φ of (2.1).

Let ω be a given measurable subset of $(0,\pi)$ of positive Lebesgue measure. The equation (2.1) is said to be *observable* on ω in time T if there exists a positive constant c such that,

$$\int_0^T\int_\omega \partial_t\varphi(t,x)^2\,dxdt \geqslant cE_a(0), \tag{2.3}$$

for all $(\varphi_0,\varphi_1) \in H_0^1(0,\pi)\times L^2(0,\pi)$, where $E_a(0) = \int_0^\pi \left(\varphi_1(x)^2 + \varphi_0'(x)^2 + a(x)\varphi_0(x)^2\right)\,dx$.

Note that in the case where $a(\cdot) = 0$, for every subset ω of $[0,\pi]$ of positive measure, it is well known that the observability inequality (2.3) is satisfied whenever $T \geqslant 2\pi$ (see [93]).

If (2.1) is observable on ω in time T, we denote by $c(T,a,\omega)$ the largest constant in (2.3), that is

$$c(T,a,\omega)=\inf_{\substack{(\varphi_0,\varphi_1)\in H_0^1(0,\pi)\times L^2(0,\pi)\\(\varphi_0,\varphi_1)\neq(0,0)}}\frac{\int_0^T\int_\omega \partial_t\varphi(t,x)^2\,dxdt}{\int_0^\pi\left(\varphi_1(x)^2+\varphi_0'(x)^2+a(x)\varphi_0(x)^2\right)\,dx}. \tag{2.4}$$

Even in the simple case $a=0$, it is not obvious to determine the constant $c(T,a,\omega)$ for arbitrary choices of ω. Indeed, a spectral expansion of the solution φ on a spectral basis shows the emergence of nontrivial crossed terms. Reformulating this question in terms of the Fourier coefficients of the initial data (φ_0,φ_1), the quantity $c(T,a,\omega)$ is seen as the optimal value of a quadratic functional over every sequence $c=(c_j)_{j\in\mathbb{Z}}\in\ell^2(\mathbb{C})$ of Fourier coefficients such that $\|c\|_{\ell^2(\mathbb{C})}=1$. This leads to a delicate mathematical problem. As it will be highlighted in the sequel, it is quite similar to the well-known open problem of determining what are the best constants in Ingham's inequalities.

This article is devoted to the introduction and description of several methods permiting to determine explicit positive constants c such that the inequality (2.3) holds for all $(\varphi_0,\varphi_1)\in H_0^1(0,\pi)\times L^2(0,\pi)$, or equivalently such that $c(T,a,\omega)\geqslant c$. One also requires that the constant c only depend on T, possibly $|\omega|$, as well as some L^∞ bounds on the potential function $a(\cdot)$. This way, as mentioned above, it is possible to deal with experiments where one only knows partial informations on the operating conditions.

It is structured as follows : the main results are presented in Section 2.1.2, as well as the presentation of each method (spectral versus propagation). In Section 2.3, we present applications of our results to control and stabilization of wave equations. Section 2.4, is devoted to the illustration of the main results and we provide several numerical simulations to comment on both methods, illustrate and compare them. For the convenience of the reader, most of the proofs are gathered in Section 2.2.

2.1.2 Main results

In this section, we present the estimates of observability constants obtained using each method. Let $\varphi\in C^0(0,T;H_0^1(0,\pi))\cap C^1(0,T;L^2(0,\pi))$ denote the solution of (2.1) with initial data $\varphi(0,\cdot)=\varphi_0(\cdot)\in H_0^1(0,\pi)$ and $\partial_t\varphi(0,\cdot)=\varphi_1(\cdot)\in L^2(0,\pi)$. In the sequel and for the sake of simplicity, the notations r_+ or r_- will respectively denote $\max_{x\in[0,\pi]} r(x)$ and $\min_{x\in[0,\pi]} r(x)$.

First method : spectral estimates. The first method makes full use of the spectral decomposition

$$\varphi(t,x)=\sum_{j=1}^{+\infty}\left(a_j\cos(\lambda_j t)+b_j\sin(\lambda_j t)\right)e_{a,j}(x), \tag{2.5}$$

where $\{e_{a,j}\}_{k\in\mathbf{N}^*}$ denotes an orthonormal Hilbert basis of $L^2(0,\pi)$ consisting of eigenfunctions of the operator $A_a=-\partial_{xx}+a(\cdot)\,\mathrm{Id}$ with Dirichlet boundary conditions, associated with the positive eigenvalues $(\lambda_j^2)_{j\in\mathbf{N}^*}$, and

$$a_j=\int_0^\pi\varphi_0(x)e_{a,j}(x)\,dx,\qquad b_j=\frac{1}{\lambda_j}\int_0^\pi\varphi_1(x)e_{a,j}(x)\,dx, \tag{2.6}$$

for every $j\in\mathbb{N}^*$.

In the following result, we provide an estimate of the observability constant $c(T,a,\omega)$ that only depends on the parameters T, $|\omega|$ and some L^∞ bounds on the potential $a(\cdot)$. It is interesting to note that no assumption is made on the topology of the set ω. However, as highlighted in Remark 3 and in the discussion ending Section 2.1.2, this approach presents some drawbacks, in particular the fact that it can only be used when the potential function $a(\cdot)$ is close to a constant function.

Theorem 1. *Consider a function $a \in L^\infty(0,\pi;\mathbb{R}_+)$ writing $a(\cdot) = \bar{a} + r(\cdot)$, where $\bar{a} \in \mathbb{R}$ and $r_- > -1$. Let ω be a measurable subset of $(0,\pi)$ of positive measure and $T(r) = \frac{2\pi}{\gamma(r)}$ where $\gamma(r) = \frac{3-r_+ + r_-}{\sqrt{4+r_-}+\sqrt{1+r_+}}$. Assume[1] that $\|r\|_\infty < \alpha_0$ where α_0 denotes the unique[2] (positive) solution of the equation*

$$|\omega| - \sin|\omega| = |\omega|\left(8\left(e^{\frac{\sqrt{2}\pi\alpha_0}{\sqrt{1-\alpha_0}}} - 1\right) + 4\pi\alpha_0\right). \tag{2.7}$$

Then, (2.1) is observable on ω for all $T > T(r)$ with $c(T,a,\omega) \geqslant C_a(T,|\omega|)$ where

$$C_a(T,|\omega|) = K_I^1(T,r)\frac{|\omega| - \sin|\omega| - 4\pi\|r\|_\infty|\omega| - 4\sqrt{|\omega|D(r)\left(\frac{|\omega|}{2} - \frac{\sin(|\omega|)}{2} - 2\pi\|r\|_\infty|\omega|\right)}}{\pi + \left(4\pi\sqrt{D(r)} + \frac{A(r)}{2} + 2\pi D\right)},$$

$$A(r) = \frac{1}{\sqrt{1+r_-}}, \quad D(r) = e^{\sqrt{2}\pi\|r\|_\infty A(r)} - 1, \quad \text{and} \quad K_I^1(T,r) = \frac{2}{\pi}\left(T - \frac{4\pi^2}{\gamma(r)^2 T}\right). \tag{2.8}$$

According to (2.7), there holds $C_a(T,|\omega|) > 0$.
Furthermore, in the particular case where $a(\cdot) = 0$, one can improve the estimate (2.8) by setting

$$C_0(T,|\omega|) = \left[\frac{T}{2\pi}\right](|\omega| - \sin|\omega|), \tag{2.9}$$

where the bracket notation stands for the integer floor.

The following remarks are in order.

Remark 1. The assumption $r_- > -1$ sufficient when $a \in L^\infty(0,\pi;\mathbb{R}_+)$ ensures that the resolvent of the operators A_a and A_r are compact. Nevertheless, the assumptions of this theorem can be extended to potential functions that are non necessarily nonnegative. More precisely, replacing the assumption $r_- > -1$ by $\min\{r_-, r_- + \bar{a}\} > -1$ leads to the same conclusion.

Remark 2. Note that the decomposition of the potential as $a(\cdot) = \bar{a} + r(\cdot)$ where $\bar{a} \in \mathbb{R}$, $r_- \leqslant r(\cdot) \leqslant r_+$ almost everywhere in $(0,\pi)$ with $\min\{r_-, r_- + \bar{a}\} > -1$ and $\|r\|_\infty < \alpha_0$ may be non-unique whenever it holds. As a consequence, it is relevant, at least from a numerical point of view to look for the best decomposition, that is the one maximizing the estimate (2.8).

Remark 3 (Smallness of α_0)**.** The constant appearing in the statement of Theorem 1 is quite small. Indeed, by using the inequality $e^h - 1 \geqslant h$ holding for all $h \geqslant 0$, it is easy to obtain

$$\alpha_0 \leqslant \frac{1}{(8\sqrt{2}+4)\pi}\left(1 - \frac{\sin(|\omega|)}{|\omega|}\right).$$

Numerical computation leads to $\frac{1}{(8\sqrt{2}+4)\pi} < 2.08.10^{-2}$ and in particular, $\alpha_0 \leqslant 2.08.10^{-2}$.

1. With the notations of this theorem, one has $\|r\|_\infty = \max\{|r_-|, |r_+|\}$.

2. Uniqueness of α_0 follows from the fact that the mapping $F : \mathbb{R} \ni \alpha_0 \mapsto 8\left(e^{\frac{\sqrt{2}\pi\alpha_0}{\sqrt{1-\alpha_0}}} - 1\right) + 4\pi\alpha_0$ is increasing.

Remark 4 (Simplifying the condition (2.7))**.**
In the statement of Theorem 1, the condition $\|r\|_\infty < \alpha_0$ is fulfilled as soon as

$$\|r\|_\infty \leqslant \beta_0 \left(1 - \frac{\sin(|\omega|)}{|\omega|}\right),$$

with $\beta_0 = 1.93.10^{-2}$.

The value of β_0 can be easily computed by using the inequality $e^h - 1 \leqslant h + h^2$ whenever $0 \leqslant h \leqslant 1$. This inequality leads to

$$\frac{1}{\beta_0} \leqslant 4\pi \left(1 - \frac{2\sqrt{2}}{\sqrt{1-\underline{\alpha}}} + \frac{4\pi\underline{\alpha}}{1-\underline{\alpha}}\right),$$

where $\underline{\alpha} = \frac{1}{(8\sqrt{2}+4)\pi} < 2.08.10^{-2}$ and the conclusion follows.

This means that Theorem 1 holds only for very small variations around constant potentials. The next theorem provides an estimate holding for a much larger class of potentials. Nevertheless, as illustrated in Section 2.4, the interest of the estimate given in Theorem 1 rests upon the fact that it is sharper whenever it can be applied. In particular, we will see that for particular resonant observability times and the choice $a(\cdot) = 0$, it coincides with the value of the observability constant $c(T, a, \omega)$.

Remark 5. The constant $K_I^1(T, a)$ introduced in the statement of Theorem 1 is a so-called Ingham constant, first introduced by Ingham in [53]. Ingham's inequality constitutes a fundamental result in the theory of nonharmonic Fourier series. It asserts that, for every real number γ and every $T > \frac{2\pi}{\gamma}$, there exist two positive constants $C_1(T,\gamma)$ and $C_2(T,\gamma)$ such that for every sequence of real numbers $(\mu_n)_{n\in\mathbb{N}^*}$ satisfying

$$\forall n \in \mathbb{N}^* \quad |\mu_{n+1} - \mu_n| \geqslant \gamma,$$

there holds

$$C_1(T,\gamma) \sum_{n\in\mathbb{Z}^*} |a_n|^2 \leqslant \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n e^{i\mu_n t} \right|^2 dt \leqslant C_2(T,\gamma) \sum_{n\in\mathbb{Z}^*} |a_n|^2, \tag{2.10}$$

for every $(a_n)_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{C})$. Denoting by $C_1(T,\gamma)$ and $C_2(T,\gamma)$ the optimal constants in (2.10), several explicit estimates of these constants are provided in [53]. For example, it is proved in this article that

$$C_1(T,\gamma) \geqslant \frac{2}{\pi}\left(T - \frac{4\pi^2}{\gamma^2 T}\right) \quad \text{and} \quad C_2(T,\gamma) \leqslant \frac{10T}{\pi}.$$

Notice that, up to our knowledge, the best constants in [53] are not known. In the particular case where $\mu_n = n$ for every $n \in \mathbb{N}^*$, one shows easily that for every $T > 2\pi$, $C_1(T,\gamma) = \left[\frac{T}{2\pi}\right]\pi$ and $C_2(T,\gamma) = C_1(T,\gamma) + 1$, the bracket notation standing for the integer floor.

In a general way, one could choose $K_I^1(T, r) = C_1(T, \gamma(r))$ with $T > \frac{2\pi}{\gamma(r)}$ in the statement of Theorem 1.

Finally, let us mention that the idea to use Ingham inequalities in control theory is a long story (see for instance [14, 43, 54, 55, 62]).

Remark 6. Notice that, due to our use of Ingham's inequality, the time $T(r)$ needed to apply Theorem 1 is greater than the minimal time of observability (see e.g. [15] for the computation of such a time). This restriction is proper to the use of spectral methods. Indeed, even in the very simple case where the potential $a(\cdot)$ vanishes, the time $T(r)$ is equal to 2π and is then greater than the minimal observability time. It should be possible to decrease the time $T(r)$ by using only the asymptotic spectral gap (see e.g. [54]), but our main interest here concerns the observability constant and the methods relying on asymptotic gaps do not usually provide good estimates. Actually, when T approaches the minimal observabiity time, the observability constant tends to 0. In particular it is not good to be close to the minimal time when we look for a sharp decay estimate of solutions to the equation with dissipative feedback control.

Particular examples of application of this theorem to observation, control and stabilization of one dimensional wave equations are provided in Section 2.4.

Second method : a propagation argument. This method makes great use of propagation properties of the wave equation to derive sharp energy estimates and is inspired by [45]. The technique consists in inverting the roles of the time and space variables, and to propagate the information from the observation domain to other ones. Although the result presented in the next theorem appears a bit technical, the approach used here is robust and holds for very large choices of potential functions $a(\cdot)$, as it will be commented in Section 2.4. Note that a close but non-quantitative result has been obtained in [99, Theorem 4] for semilinear wave equations. In the next result, we keep track of the constants in this method, trying to improve each step by choosing the best possible parameters (see for instance Lemma 5 or Remark 11 in the proof).

In the next result and unlike the framework of Theorem 1, the estimate of the observability constant $c(T, a, \omega)$ only depends on T, $\|a\|_\infty$ and the precise knowledge of ω.

Theorem 2. *Let $\omega = (\alpha, \beta)$ with $0 < \alpha < \beta < \pi$ and $T_0 = 2\max\{\alpha, \pi - \beta\}$. Define for $\eta > 0$ and $k \geqslant 0$, the quantity*

$$K(\eta, \alpha, \beta, k) = \begin{cases} C(\eta)\left(\frac{e^{2k(\pi-\beta+3\eta)}-e^{4k\eta}+e^{-2k\eta}-e^{-2k(\alpha+2\eta)}}{2k\eta}+1\right) & \text{if } k > 0 \\ C(\eta)\left(\frac{\pi-\beta+\alpha}{\eta}+3\right) & \text{if } k = 0 \end{cases} \tag{2.11}$$

where $C(\eta) = \frac{1}{2\eta} + \max\left\{1, \frac{1}{\eta^2}\right\} + \max\{1, k^2\}$. Then, (2.1) is observable on ω for all $T > T_0$ with $c(T, a, \omega) \geqslant C'_{T,T_0,a,\lambda}(\alpha, \beta)$ where

— *if $T \in (T_0, 2T_0)$, then*

$$C'_{T,T_0,a,\lambda}(\alpha, \beta) = \max_{(\gamma,\eta)\in\mathcal{A}_{\alpha,\beta,T_0,T}} \left(\frac{\gamma}{2} - \sup_{j\in\mathbf{N}^*} \frac{|\sin(\lambda_j\gamma)|}{2\lambda_j}\right) \frac{4(T-T_0-2\gamma)}{(8+T^2)\gamma K(\eta, \alpha, \beta, \|a\|_\infty^{1/2})} \tag{2.12}$$

and $\mathcal{A}_{\alpha,\beta,T_0,T} = \left\{(\gamma, \eta) \mid \gamma \in \left(0, \frac{T-T_0}{4}\right) \text{ and } \eta \in \left(0, \min\left\{\frac{T-T_0}{16} - \frac{\gamma}{8}, \frac{\beta-\alpha}{4}\right\}\right)\right\}$,

— *if $T \geqslant 2T_0$, then*

$$C'_{T,T_0,a,\lambda}(\alpha, \beta) = \left[\frac{T}{2T_0}\right] \max_{(\gamma,\eta)\in\mathcal{A}_{\alpha,\beta,T_0,T}} \left(\frac{\gamma}{2} - \sup_{j\in\mathbf{N}^*} \frac{|\sin(\lambda_j\gamma)|}{2\lambda_j}\right) \frac{T_0-2\gamma}{(2+T_0^2)\gamma K(\eta, \alpha, \beta, \|a\|_\infty^{1/2})} \tag{2.13}$$

and $\mathcal{A}_{\alpha,\beta,T_0,T} = \left\{(\gamma, \eta) \mid \gamma \in \left(0, \frac{T_0}{4}\right) \text{ and } \eta \in \left(0, \min\left\{\frac{T_0}{16} - \frac{\gamma}{8}, \frac{\beta-\alpha}{4}\right\}\right)\right\}$.

Remark 7. We stress the fact that, unlike the statement of Theorem 1, no restriction on the L^∞ norm of the function $a(\cdot)$ is needed in the statement of Theorem 2.

Remark 8. The constant $C'_{T,a,\lambda}(\alpha, \beta)$ given by (2.12) or (2.13) writes as the maximum of a two variables function. Due to the presence of highly nonlinear terms in its expression, the maximum cannot be computed explicitly in general, but is nevertheless easy to compute numerically. It will be illustrated in Section 2.4.

Remark 9. Notice that, in the case $a(\cdot) = 0$, we have $\lambda_j = j$ for every $j \in \mathbb{N}^*$ and the quantity $\frac{\gamma}{2} - \sup_{j\in\mathbb{N}^*} \frac{|\sin(\lambda_j\gamma)|}{2\lambda_j}$ simplifies to $\frac{\gamma}{2} - \frac{|\sin(\gamma)|}{2}$.

In the general case and to avoid to use the knowledge of the whole spectrum, one can simplify (2.12) and (2.13) by noting that

$$\sup_{j\in\mathbb{N}^*} \frac{|\sin(\lambda_j\gamma)|}{2\lambda_j} \leqslant \frac{\gamma}{2} \sup_{\{x\geqslant\lambda_1\gamma\}} \frac{|\sin x|}{x}.$$

Remark 10 (Key ingredient of the proof.)**.** The proof of Theorem 2 derives benefit from the propagation properties of Equation (2.1) along the characteristics. This is illustrated on Figure 2.1, representing the propagation of wavefronts in the time-space domain $(0, \pi) \times (0, T)$ in the case where the observation domain is $\omega = (\alpha, \beta)$. Recall that every point $x_0 \in (0, \pi)$ generates two characteristics : one is going to the left and the other one to the right, in a symmetrical way. Roughly speaking, the solution φ of the wave equation (2.1) is known on the light-gray rectangle domain, and the propagation properties of the wave equation allow to recover φ on the the deep-gray domain, provided that the observation time T be large enough.

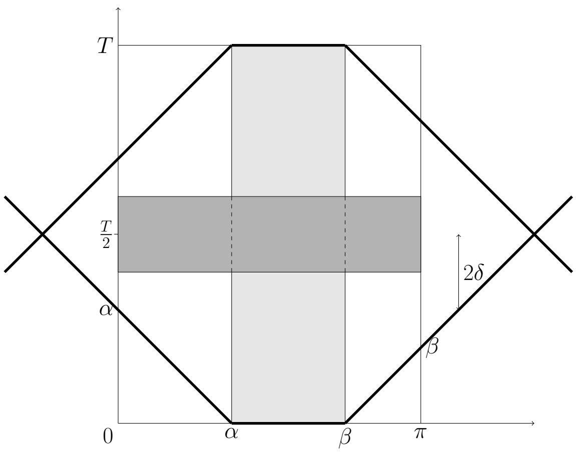


FIGURE 2.1 – Propagation zones along the characteristics

As a corollary of Theorem 2, we have the following result, extending the estimate of the observability constant to those subsets ω that are the finite union of open intervals.

Corollary 1. *Let $\omega = \bigcup_{i=1}^n (\alpha_i, \beta_i)$ with $0 < \alpha_1 < \beta_1 < \cdots < \alpha_n < \beta_n < \pi$ and $T_0 = 2\max_{1\leqslant i\leqslant n}\{\alpha_i - \beta_{i-1}, \alpha_{i+1} - \beta_i\}$, with the convention that $\beta_0 = 0$ and $\alpha_{n+1} = \pi$. Define*

$$K'(\eta_i, k) = \begin{cases} C(\eta_i)\left(\frac{e^{2k(\alpha_{i+1}-\beta_i+3\eta_i)} - e^{4k\eta_i} + e^{-2k\eta_i} - e^{-2k(\beta_{i-1}-\alpha_i+2\eta_i)}}{2k\eta_i} + 1\right) & \text{if } k > 0 \\ C(\eta_i)\left(\frac{\alpha_{i+1}-\beta_i+\alpha_i-\beta_{i-1}}{\eta_i} + 3\right) & \text{if } k = 0 \end{cases} \tag{2.14}$$

for $i \in \{1, \ldots, n\}$, $\eta_i > 0$ and $k \geqslant 0$. Then, we have $c(T, a, \omega) \geqslant \min_{1\leqslant i\leqslant n} C'_{T,T_0,a,\lambda}(\alpha_i, \beta_i)$, where the quantity $K(\eta_i, \alpha_i, \beta_i, k)$ defined by (2.11) has been replaced by the quantity $K'(\eta_i, k)$ defined by (2.14) in the definition of the constant $C'_{T,T_0,a,\lambda}(\alpha_i, \beta_i)$.

In Theorem 2 and Corollary 1, the time T_0 coincides with the smallest observability time. The following corollary provides a simpler estimate of the observability constant, but is only valid for potentially larger values of the observation time T and provided that we have a precise knowledge of the low frequencies $(\lambda_j)_{j\in\mathbb{N}^*}$ associated to the operator $-\partial_{xx}+a(\cdot)\,\mathrm{Id}$, introduced at the beginning of this section. The proof is based on Ingham inequalities [53].

Corollary 2. *Let $\omega=\bigcup_{i=1}^n(\alpha_i,\beta_i)$ with $0<\alpha_1<\beta_1<\cdots<\alpha_n<\beta_n<\pi$. Let $j_0=\left[\frac{\|a\|_\infty-1}{2}\right]+1$ and introduce the positive real numbers*

$$T_0'=\frac{2\pi}{\gamma_{j_0}}\quad\textit{and}\quad \gamma_{j_0}=\begin{cases}\frac{1+2j_0-\|a\|_\infty}{j_0+1+\sqrt{j_0^2+\|a\|_\infty}} & \textit{if } j_0=1\\ \min\left\{\min_{j\in\{1\dots j_0-1\}}\lambda_{j+1}-\lambda_j,\frac{1+2j_0-\|a\|_\infty}{j_0+1+\sqrt{j_0^2+\|a\|_\infty}}\right\} & \textit{if } j_0>1\end{cases}.$$

Then, we have

$$c(T,a,\omega)\geqslant \widetilde{C}_{T,T_0',a,\lambda}(\omega)=\frac{(T-T_0')(\gamma_{j_0}^2T^2-4\pi^2)}{2(\gamma_{j_0}^2T^2-4\pi^2)+10\gamma_{j_0}^2T^2}\Lambda_{T,T_0',a}(\omega)\tag{2.15}$$

with

$$\Lambda_{T,T_0',a}(\omega)=\min_{1\leqslant i\leqslant n}\max\left\{\frac{1}{K(\eta,\alpha_i,\beta_i,\|a\|_\infty^{1/2})},0<\eta<\min\left\{\frac{T-T_0'}{16},\frac{\beta_i-\alpha_i}{4}\right\}\right\}.$$

Short discussion on the main results. We stress the fact that the main ingredients in the proofs of Theorem 1 on the first hand, and Theorem 2, Corollaries 1 and 2 are of different natures. Indeed, Theorem 1 is based on a purely spectral argument, and is somehow limited by the misreading of the low frequencies of the spectrum. This explains the restrictions on the norm of the potential $\|a\|_\infty$. By the way, notice that the knowledge of the value of a spectral gap γ (in the sense made precise in Remark 5) would permit to avoid the technicalities in Lemmas 2, 3 and 4 and to get directly a simple estimate instead of the one of Theorem 2. The results in Theorem 2, Corollaries 1 and 2 are in some sense more robust since we do not need to make assumptions on the smallness of the difference between the maximal and minimal values of the potential $a(\cdot)$. Moreover, on the contrary to the spectral method, propagation ones provide a lower bound of the observability constant working when the observation time T is close to the minimal observation time T_0 when ω is an interval.

Recall also that, independently of the aforementioned drawbacks, in the context of an inverse problem where there is some indetermination on some parameters of the problem, Theorem 1 can be used when the topological nature (in particular the number of connected components) of ω is not known whereas Theorem 2 assumes that ω is known and writes as a finite union of intervals. In each case, one can assume that only L^∞ bounds are known on the potential $a(\cdot)$.

To sum-up, we gather this discussion under the following condensed form.

	requires the knowledge of	Advantage	Drawback
Spectral method	T, $\lvert\omega\rvert$, L^∞ bounds on $a(\cdot)$	- sharp estimate, - ω is only assumed to be measurable	works for almost constant potentials
Propagation method	T, ω, L^∞ bounds on $a(\cdot)$	no restriction on the potentials	- estimate not so accurate - ω writes as a finite union of intervals

Section 2.4 is devoted to the numerical comparison and results obtained using each method.

2.2 Proofs of the main results

2.2.1 Proof of Theorem 1

Let us begin by proving the second part of the statement of Theorem 1, the general case using strongly the estimate obtained in this simple case.

The case $a(\cdot) = 0$. Assume that $a(\cdot) = 0$. According to (2.5) and since the eigenfunctions of the Dirichlet-Laplacian operator on $\Omega = (0, \pi)$ are given by $e_{0,j}(x) = \sqrt{\frac{2}{\pi}}\sin(jx)$ for every $j \in \mathbb{N}^*$ it follows that for all initial data $(\varphi^0, \varphi^1) \in H_0^1(0, \pi) \times L^2(0, \pi)$, the solution $\varphi \in C^0(0, T; H_0^1(0, \pi)) \cap C^1(0, T; L^2(0, \pi))$ of (2.1) can be expanded as

$$\varphi(t, x) = \sum_{j=1}^{+\infty} (a_j \cos(jt) + b_j \sin(jt)) \sin(jx), \tag{2.16}$$

where the sequences $(ja_j)_{j\in\mathbb{N}^*}$ and $(jb_j)_{j\in\mathbb{N}^*}$ belong to $\ell^2(\mathbb{R})$ and are determined in function of the initial data (φ^0, φ^1) by

$$a_j = \frac{2}{\pi}\int_0^\pi \varphi^0(x) \sin(jx)\, dx, \quad b_j = \frac{2}{j\pi}\int_0^\pi \varphi^1(x) \sin(jx)\, dx, \tag{2.17}$$

for every $j \in \mathbb{N}^*$.

By the way, note that

$$\|(\varphi^0, \varphi^1)\|^2_{H_0^1\times L^2} = \frac{\pi}{2}\sum_{j=1}^{+\infty} j^2(a_j^2 + b_j^2), \tag{2.18}$$

and furthermore, one has

$$\begin{aligned}
\int_0^T\int_\omega |\partial_t\varphi(t,x)|^2\, dxdt &\geqslant \int_\omega\int_0^{2\pi\left[\frac{T}{2\pi}\right]} |\partial_t\varphi(t,x)|^2\, dtdx \\
&= \left[\frac{T}{2\pi}\right]\int_\omega\int_0^{2\pi}\left|\sum_{j=1}^{+\infty} j(-a_j\sin(jt) + b_j\cos(jt))\sin(jx)\right|^2 dtdx \\
&= \pi\left[\frac{T}{2\pi}\right]\sum_{j=1}^{\infty} j^2(a_j^2+b_j^2)\int_\omega \sin^2(jx)\, dx.
\end{aligned}$$

The following lemma is a crucial ingredient to conclude.

Lemma 1. *Let $j \in \mathbb{N}^*$. For every measurable subset ω of $(0,\pi)$, one has*

$$\int_\omega \sin(jx)^2\, dx \geqslant \frac{|\omega| - \sin|\omega|}{2}.$$

This lemma is noticed as well in [82, 85] and used for controllability purposes in [63]. Even though it is well known, we provide at the end of this paragraph an elementary and new proof of this result using the Schwarz symmetrization.

As a consequence, one gets

$$\int_0^T \int_\omega |\partial_t \varphi(t,x)|^2\, dxdt \geqslant \frac{\pi}{2}\left[\frac{T}{2\pi}\right](|\omega| - \sin|\omega|)\sum_{j=1}^\infty j^2(a_j^2+b_j^2).$$

Combining this inequality with (2.18), we infer

$$\left[\frac{T}{2\pi}\right](|\omega| - \sin|\omega|)\int_0^\pi \left(\varphi_1(x)^2 + \varphi_0'(x)^2\right)\, dx \leqslant \int_0^T\int_\omega \partial_t\varphi(t,x)^2\, dxdt,$$

whence the estimate (2.9).

Proof of Lemma 1. Let us first estimate the quantity

$$\delta_j(L) = \sup\left\{\int_\omega f_j(x)\, dx,\ \omega \text{ measurable subset of } (0,\pi) \text{ such that } |\omega| = L\pi\right\},$$

where $f_j(x) = \sin(jx)^2$ and $L \in (0,1)$ is fixed. Let ω be a measurable subset of $(0,\pi)$ such that $|\omega| = L\pi$. Denote by $f_{j,S}$ the Schwarz rearrangement [3] of the function f_j. According to the Hardy-Littlewood inequality (see e.g. [49, 59]), one has

$$\int_\omega f_j(x)\, dx = \int_0^\pi \chi_\omega(x) f_j(x)\, dx \leqslant \int_0^\pi \chi_{\omega_S}(x) f_{j,S}(x)\, dx = \int_{\omega_S} f_{j,S}(x)\, dx.$$

Moreover, notice that $f_{j,S} = f_1$ for every $j \in \mathbb{N}^*$. Indeed, introducing for every $\mu \in (0,1)$ and $j \in \mathbb{N}^*$ the set $\Omega_j(\mu) = \{x \in (0,\pi) \mid f_j(x) \geqslant \mu\}$, a simple computation ensures that

$$\Omega_j(\mu) = \bigcup_{k=1}^j \left\{x \in (0,\pi) \mid \left|x - \frac{(2k-1)\pi}{2j}\right| \leqslant \frac{(2k-1)\pi}{j} - \frac{\arcsin(\sqrt{\mu})}{j}\right\},$$

so that, $|\Omega_j(\mu)| = \pi - 2\arcsin(\sqrt{\mu}) = |\Omega_1(\mu)|$. The expected result follows easily.

As a consequence and since $\omega_S = ((1-L)\pi/2, (1+L)\pi/2)$, one has

$$\int_\omega f_j(x)\, dx \leqslant \int_{\omega_S} f_1(x)\, dx = \frac{L\pi + \sin(L\pi)}{2}.$$

3. For every subset U of Ω, we denote by U_S the ball centered at $\frac{\pi}{2}$ having the same Lebesgue measure as U. We recall that, for every nonnegative Lebesgue measurable function u defined on Ω and vanishing on its boundary, denoting by $\Omega(c) = \{x \in \Omega \mid u(x) \geqslant c\}$ its level sets, the Schwarz rearrangement of u is the function u_S defined on Ω_S by

$$u_S(x) = \sup\{c \mid x \in (\Omega(c))_S\}.$$

The function u_S is built from u by rearranging the level sets of u into balls having the same Lebesgue measure (see, e.g., [49, Chapter 2]).

and then $\delta_j(L) \leqslant \frac{L\pi+\sin(L\pi)}{2}$ for every $j \in \mathbb{N}^*$. Finally, denoting by ω^c the complement set of ω in $(0,\pi)$, we get the expected inequality by writing

$$\begin{aligned}\inf_{\substack{\omega \text{ measurable}\\ |\omega|=L\pi}} \int_\omega \sin(jx)^2\,dx &= \frac{\pi}{2} - \sup_{\substack{\omega^c \text{ measurable}\\ |\omega^c|=(1-L)\pi}} \int_{\omega^c} \sin(jx)^2\,dx \\ &\geqslant \frac{\pi}{2} - \left(\frac{(1-L)\pi + \sin((1-L)\pi)}{2}\right) = \frac{L\pi - \sin(L\pi)}{2}.\end{aligned}$$

We thus infer that

$$\int_\omega \sin(jx)^2\,dx \geqslant \frac{|\omega| - \sin|\omega|}{2}$$

for every measurable subset ω of $(0,\pi)$. $\square$

The case $\bar{a} = 0$. We start with some elementary lemmas that play an important role.

Lemma 2. *Assume that $r_- > -1$ and $r_+ - r_- < 3$. Then, for every $j \in \mathbb{N}^*$, one has*

$$\lambda_{j+1} - \lambda_j \geqslant \frac{3 - r_+ + r_-}{\sqrt{4+r_-} + \sqrt{1+r_+}}.$$

Démonstration. The Courant-Fischer minimax principle writes

$$\lambda_j^2 = \min_{\substack{V\subset H_0^1(0,\pi)\\ \dim V = j}} \max_{u\in V\setminus\{0\}} \frac{\int_0^\pi (u'(x)^2 + r(x)u(x)^2)dx}{\int_0^\pi u(x)^2\,dx}.$$

Using that $r_- \leqslant r(x) \leqslant r_+$ for almost every $x \in (0,\pi)$ yields

$$\sqrt{j^2 + r_-} \leqslant \lambda_j \leqslant \sqrt{j^2 + r_+}, \tag{2.19}$$

for every $j \in \mathbb{N}^*$. It suffices indeed to compare λ_j^2 with the j-th eigenvalue of a Sturm-Liouville operator with constant coefficients. We infer

$$\begin{aligned}\lambda_{j+1} - \lambda_j &\geqslant \sqrt{(j+1)^2 + r_-} - \sqrt{j^2 + r_+}, \\ &= \frac{1 + 2j - r_+ + r_-}{\sqrt{(j+1)^2 + r_-} + \sqrt{j^2 + r_+}},\end{aligned}$$

for every $j \in \mathbb{N}^*$. The sequence $j \mapsto \sqrt{(j+1)^2 + r_-} - \sqrt{j^2 + r_+}$ being increasing, the expected estimate follows. $\square$

Notice that similar computations on the asymptotic spectral gap of Sturm-Liouville operators may be found in [57].

Lemma 3. *Let $j \in \mathbb{N}^*$, $r \in L^\infty(0,\pi)$ such that $r_- > -1$ and ω be a measurable subset of $(0,\pi)$. The j-th eigenfunction $e_{r,j}(\cdot)$ of the operator $A_r = -\partial_{xx} + r(\cdot)\,\mathrm{Id}$ satisfies*

$$e_{r,j}(x)^2 \geqslant \frac{2}{C_j}\left(\sin(\lambda_j x)^2 + 2\lambda_j h_j(x)\sin(\lambda_j x)\right) \tag{2.20}$$

for every $x\in[0,\pi]$, *where*

$$C_j=\pi+\frac{2}{\lambda_j}\left(2\pi\sqrt{(1+r_-)\left(e^{\tau(r)\pi}-1\right)}+\frac{1}{4}+\frac{\pi(1+r_-)\left(e^{\tau(r)\pi}-1\right)}{\lambda_j}\right), \tag{2.21}$$

$h_j\in L^\infty(0,\pi)$ *is such that* $\|h_j\|_\infty^2\leqslant\frac{(1+r_-)(e^{\tau(r)\pi}-1)}{\lambda_j^4}$ *and* $\tau(r)=\frac{\sqrt{2}\|r\|_\infty}{\sqrt{1+r_-}}$.

Démonstration. The estimate (2.20) is obtained by using a shooting method that we roughly describe. Introduce $\psi_j:x\mapsto\frac{\sin(\lambda_j x)}{\lambda_j}$. The function $\phi_j=\frac{e_{r,j}(\cdot)}{e'_{r,j}(0)}$ solves the following Cauchy system

$$\begin{array}{l}-\phi_j''(x)+r(x)\phi_j(x)=\lambda_j^2\phi_j(x)\quad x\in(0,\pi),\\ \phi_j(0)=0,\quad \phi_j'(0)=1,\end{array}$$

whereas the function $h_j=\phi_j-\psi_j$ solves the Cauchy system

$$\begin{array}{l}-h_j''(x)+(r(x)-\lambda_j^2)h_j(x)=-r(x)\psi_j(x)\quad x\in(0,\pi),\\ h_j(0)=0,\quad h_j'(0)=0.\end{array} \tag{2.22}$$

Let us denote by $\mathcal{V}_{\lambda_j}$ the energy function defined by

$$\mathcal{V}_{\lambda_j}(x)=\frac{1}{2}h_j'^2(x)+\frac{\lambda_j^2}{2}h_j^2(x).$$

Using (2.22), one gets the estimate

$$\mathcal{V}_{\lambda_j}'(x)\leqslant\tau(r)\mathcal{V}_{\lambda_j}(x)+\frac{\|r\|_\infty^2}{\tau(r)\lambda_j^2},$$

for every $x\in[0,\pi]$, with $\tau(r)=\frac{\sqrt{2}\|r\|_\infty}{\sqrt{1+r_-}}$. According to the Gronwall lemma, we infer

$$\|h_j\|_\infty^2\leqslant\frac{1+r_-}{\lambda_j^4}\left(e^{\tau(r)\pi}-1\right).$$

As a consequence, and since $e_{r,j}(\cdot)=\frac{\phi_j(\cdot)}{\sqrt{\int_0^\pi\phi_j(x)^2\,dx}}$, one gets

$$\int_0^\pi\phi_j(x)^2\,dx=\int_0^\pi\left(\frac{\sin^2(\lambda_j x)}{\lambda_j^2}+2h_j(x)\frac{\sin(\lambda_j x)}{\lambda_j}+h_j(x)^2\right)dx\leqslant\frac{C_j}{2\lambda_j^2},$$

where C_j is defined by (2.21). The expected estimate then follows. □

Lemma 4. *Let* $r\in L^\infty(0,\pi)$ *such that* $r_->-1$. *Then, one has*

$$\int_\omega\sin^2(\lambda_j x)\,dx\geqslant\frac{|\omega|}{2}-\frac{\sin|\omega|}{2}-\frac{2\pi\|r\|_\infty|\omega|}{j},$$

for every $j\in\mathbb{N}^*$.

Démonstration. In accordance with (2.19), let us write $\lambda_j=j+\frac{\ell_j}{j}$ with $r_-\leqslant l_j\leqslant r_+$ for every

$j \in \mathbb{N}^*$. Using Lemma 1, we get

$$
\begin{aligned}
\int_\omega \sin^2(\lambda_j x)dx &= \frac{|\omega|}{2} - \frac{1}{2}\int_\omega \cos(2\lambda_j x)\,dx, \\
&= \int_\omega \left(\sin^2(jx) + \frac{\ell_j}{j}\cos(2jx)\int_0^x \sin\left(\frac{2s\ell_j}{j}\right)ds + \frac{1}{2}\sin(2jx)\sin\left(\frac{2\ell_j x}{j}\right)\right)dx, \\
&\geqslant \frac{|\omega|}{2} - \frac{\sin|\omega|}{2} - \frac{2\pi|l_j||\omega|}{j}.
\end{aligned}
$$

Since $|l_j| \leqslant \|r\|_\infty$,

$$
\int_\omega \sin^2(\lambda_j x)dx \geqslant \frac{|\omega|}{2} - \frac{\sin|\omega|}{2} - \frac{2\pi\|r\|_\infty|\omega|}{j}.
$$

□

We now prove Theorem 1. Combining (2.20) with the Cauchy-Schwarz inequality, one gets

$$
\int_\omega e_{r,j}(x)^2\,dx \geqslant \frac{2}{C_j}\left(\int_\omega \sin^2(\lambda_j x)\,dx - 2\lambda_j\sqrt{\int_\omega h_j(x)^2dx}\sqrt{\int_\omega \sin^2(\lambda_j x)\,dx}\right) \tag{2.23}
$$

for every $j \in \mathbb{N}^*$. The positivity of the right hand side in the inequality above is equivalent to the condition $\int_\omega \sin^2(\lambda_j x)\,dx > 4\lambda_j^2\int_\omega h_j(x)^2dx$. Moreover, according to Lemma 3, one shows easily that this condition holds whenever $\|r\|_\infty < \alpha_0$, where α_0 denotes the unique solution of (2.7) with $\bar{a} = 0$. Finally, the constant C_j being defined by (2.21), one finally gets using the estimate (2.19) an upper bound on it, uniform with respect to j, so that

$$
\int_\omega e_{r,j}(x)^2\,dx \geqslant \frac{|\omega| - \sin|\omega| - 4\pi\|r\|_\infty|\omega| - 4\sqrt{|\omega|D(r)\left(\frac{|\omega|}{2} - \frac{\sin(|\omega|)}{2} - 2\pi\|r\|_\infty|\omega|\right)}}{\pi + \left(4\pi\sqrt{D(r)} + \frac{A(r)}{2} + 2\pi D(r)\right)}, \tag{2.24}
$$

for every $j \in \mathbb{N}^*$, using the notations introduced in the statement of Theorem 1.

By decomposing the solution φ of (2.1) in the spectral basis $\{e_{r,j}\}_{j\in\mathbb{N}^*}$ as in (2.5)-(2.6), one gets using the so-called Ingham inequality

$$
\begin{aligned}
\int_0^T\int_\omega |\partial_t\varphi(t,x)|^2\,dxdt &= \frac{1}{2}\int_0^T\int_\omega\left|\sum_{k\in\mathbb{Z}^*} i\,\mathrm{sgn}(k)\lambda_{|k|}\sqrt{a_{|k|}^2 + b_{|k|}^2}\,e^{i\,\mathrm{sgn}(k)(\lambda_{|k|}t - \theta_{|k|})}e_{r,|k|}(x)\right|^2 dtdx, \\
&\geqslant \frac{K_I^1(T,a)}{2}\sum_{k\in\mathbb{Z}^*}(a_{|k|}^2 + b_{|k|}^2)\lambda_{|k|}^2\int_\omega e_{r,|k|}(x)^2\,dx \\
&= K_I^1(T,a)\sum_{j=1}^{+\infty}(a_j^2 + b_j^2)\lambda_j^2\int_\omega e_{r,j}(x)^2\,dx,
\end{aligned}
$$

where $(\theta_j)_{j\in\mathbb{N}^*}$ denotes the sequence defined by $e^{i\theta_j} = \frac{a_j + ib_j}{\sqrt{a_j^2 + b_j^2}}$ for every $j \in \mathbb{N}^*$, and $K_I^1(T,a)$ is the Ingham constant introduced in the statement of Theorem 1 and whose choice is commented in Remark 5.

The energy identity

$$\int_0^\pi \left(\partial_t \varphi^2(t,x) + \partial_x \varphi^2(t,x) + a(x)\varphi^2(t,x)\right)\, dx = \sum_{j=1}^{+\infty} \lambda_j^2 (a_j^2 + b_j^2),$$

holding for almost every $t \in [0,T]$, we infer

$$\int_0^T \int_\omega |\partial_t \varphi(t,x)|^2 \, dxdt \geqslant K_I^1(T,a) \inf_{j\in \mathbb{N}^*} \int_\omega e_{r,j}(x)^2 \, dx \int_0^\pi \left(\partial_t \varphi^2(t,x) + \partial_x \varphi^2(t,x) + a(x)\varphi^2(t,x)\right)\, dx.$$

Finally, the combination of this inequality with (2.24) provides the desired result. Furthermore, by construction, the right-hand side of the obtained inequality, denoted $C_a(T, |\omega|)$ is positive.

The general case. The general estimates are obtained by mimicking the proof in the case where $\bar a = 0$. Indeed, note that λ_j^2 is an eigenvalue of the operator A_a if, and only if $\mu_j^2 = \lambda_j^2 - \bar a$ is an eigenvalue of the operator A_r. Since $\mu_j^2 \geqslant \sqrt{1 + r_-} > 0$, one has

$$\mu_{j+1} - \mu_j = \frac{\lambda_{j+1}^2 - \lambda_j^2}{\sqrt{\lambda_{j+1}^2 - \bar a} + \sqrt{\lambda_j^2 - \bar a}} \geqslant \frac{3 - (r_+ - r_-)}{\sqrt{4 + r_-} + \sqrt{1 + r_+}}$$

for every $j \in \mathbb{N}^*$. We obtain successively the same estimates as those in the statements of Lemmas 3 and 4, replacing the quantity λ_j by μ_j. The expected conclusion follows.

2.2.2 Proof of Theorem 2

For the sake of simplicity, the notations φ_x or φ_t will respectively denote the partial derivatives $\partial_x \varphi$ and $\partial_t \varphi$ in the section 2.2.2. Let us denote by φ the unique solution of the wave equation (2.1) with a given potential $a(\cdot) \in L^\infty(0,\pi)$ and with initial data $(\varphi(0,\cdot), \varphi_t(0,\cdot)) = (\varphi_0, \varphi_1) \in H_0^1(0,\pi) \times L^2(0,\pi)$.

Using the notations introduced in Section 2.1, let us define the function $\hat E_a$ by

$$\begin{array}{rccl} \hat E_a : & [0,T] \times [0,\pi] & \longrightarrow & \mathbb{R}_+ \\ & (t,x) & \longmapsto & \varphi_t(t,x)^2 + \varphi_x(t,x)^2 + a(x)\varphi(t,x)^2, \end{array}$$

and the function $\hat E_{k^2}$ by

$$\begin{array}{rccl} \hat E_{k^2} : & [0,T] \times [0,\pi] & \longrightarrow & \mathbb{R}_+ \\ & (t,x) & \longmapsto & \varphi_t(t,x)^2 + \varphi_x(t,x)^2 + k^2\varphi(t,x)^2, \end{array}$$

where $k^2 = \|a\|_\infty$.

This proof is divided into several steps and is illustrated on Figure 2.2. Let us comment on this figure : in a nutshell, the main ingredients of the proof are contained in the statements of Lemmas 6 and 7, that take advantage of the propagation properties of the wave equation (2.1) to derive energy estimates on the zone P_1 (respectively P_2) from the observation on the zone Q_1 (respectively Q_2). Finally, since we will establish estimates along the characteristics, the space and time variables will play symmetric roles in the algebraic computations that follows.

Let us start with several instrumental lemmas.

The first one states an observability result in the case where $\omega = \Omega = (0,\pi)$.

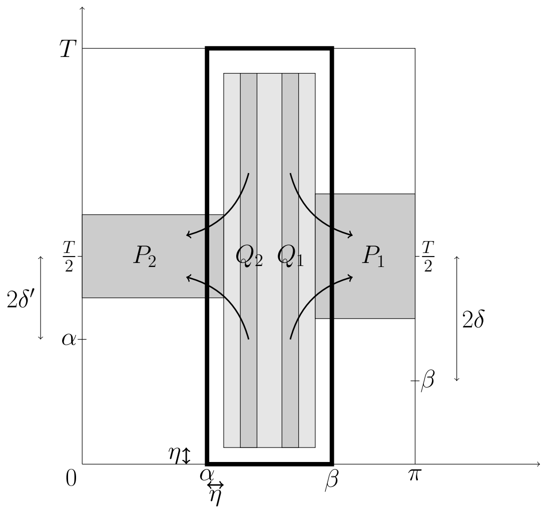


FIGURE 2.2 – Scheme of the proof (Propagation method)

Lemma 5. *Let* $\gamma \in (0, T/2)$. *Thus,*

$$\inf_{(\varphi_0,\varphi_1)\in H_0^1(0,\pi)\times L^2(0,\pi)} \frac{\int_{\frac{T}{2}-\gamma}^{\frac{T}{2}} \int_\Omega |\varphi_t(t,x)|^2\, dxdt}{\int_\Omega (\varphi_1(x)^2 + \varphi_0'(x)^2 + a(x)\varphi_0(x)^2)\, dx} = \frac{\gamma}{2} - \sup_{j\in\mathbb{N}^*} \frac{|\sin(\lambda_j\gamma)|}{2\lambda_j}, \tag{2.25}$$

where φ *denotes the unique solution of the wave equation* (2.1) *with initial data* $(\varphi(0,\cdot), \varphi_t(0,\cdot)) = (\varphi_0, \varphi_1) \in H_0^1(0,\pi) \times L^2(0,\pi)$ *and* $(\lambda_j)_{j\in\mathbb{N}^*}$ *the sequence of eigenvalues introduced in Section 2.1.2.*

Démonstration. Denote by LHS the left hand side in (2.25). Decomposing φ as in (2.5)-(2.6) yields

$$\text{LHS} = \inf_{(\lambda_j a_j,\lambda_j b_j)_{j\in\mathbb{N}^*}\in(\ell^2(\mathbb{R}))^2} \frac{\sum_{j=1}^\infty \lambda_j^2 \int_{\frac{T}{2}-\gamma}^{\frac{T}{2}} (a_j \sin(\lambda_j t) - b_j\cos(\lambda_j t))^2\, dt}{\sum_{j=1}^\infty \lambda_j^2 (a_j^2 + b_j^2)}.$$

Next, setting $\lambda_j a_j = \rho_j \cos(\theta_j)$, $\lambda_j b_j = \rho_j \sin(\theta_j)$ with $(\rho_j)_{j\in\mathbb{N}^*} \in \ell^2(\mathbb{R})$ and $\theta_j \in \mathbb{T}$ for all $j \in \mathbb{N}^*$, we get using a homogeneity argument

$$\begin{aligned}\text{LHS} &= \inf_{\sum_{j=1}^\infty \rho_j^2=1}\ \inf_{(\theta_j)_{j\in\mathbb{N}^*}} \sum_{j=1}^\infty \rho_j^2 \int_{\frac{T}{2}-\gamma}^{\frac{T}{2}} \sin^2(\lambda_j t - \theta_j)\, dt,\\ &= \frac{\gamma}{2} - \sup_{\sum_{j=1}^\infty \rho_j^2=1}\ \sup_{(\theta_j)_{j\in\mathbb{N}^*}} \sum_{j=1}^{+\infty} \rho_j^2 \frac{\sin(\lambda_j\gamma)}{2\lambda_j} \cos(\lambda_j(T-\gamma) - 2\theta_j).\end{aligned}$$

To reach the maximum, it suffices to choose θ_j such that $\cos(\lambda_j(T-\gamma) - 2\theta_j) = \text{sgn}(\sin(\lambda_j\gamma))$ for all $j \in \mathbb{N}^*$ and $(\rho_j)_{j\in\mathbb{N}^*}$ as a Kronecker delta, and we thus get (2.25). □

In the following lemma, pointed out as a main ingredient of the proof, we establish an energy estimate on $(\eta, T-\eta) \times (\alpha+\eta, \beta-\eta)$ for $\eta \in (0, \beta-\alpha)$.

Lemma 6. *Let $\eta \in (0, \beta - \alpha)$ and $T \geqslant 2\eta$. The energy estimate*

$$\int_{\eta}^{T-\eta} \int_{\alpha+\eta}^{\beta-\eta} \hat{E}_{k^2}(t,x)\, dxdt \leqslant C(\eta) \int_0^T \int_\alpha^\beta \left(\varphi_t(t,x)^2 + \varphi(t,x)^2\right)\, dxdt. \tag{2.26}$$

holds with $C(\eta) = \frac{1}{2\eta} + \max\left\{1, \frac{1}{\eta^2}\right\} + \max\{1, k^2\}$.

Démonstration. Multiplying the identity

$$(\varphi\varphi_t)_t(t,x) - (\varphi\varphi_x)_x(t,x) - \varphi_t(t,x)^2 + \varphi_x(t,x)^2 + a(x)\varphi(t,x)^2 = 0$$

valid for almost every $(t,x) \in (0,T) \times (0,\pi)$, by any smooth function $\zeta \in C_c^\infty((0,T) \times (\alpha,\beta))$ such that $\zeta \geqslant 0$ on $(0,T) \times (\alpha,\beta)$ and $\zeta = 1$ in $[\eta, T-\eta] \times [\alpha+\eta, \beta-\eta]$, we get after integration on $(0,T) \times (0,\pi)$,

$$\begin{aligned}
\int_{\alpha+\eta}^{\beta-\eta} \int_{\eta}^{T-\eta} \varphi_x(t,x)^2 \zeta(t,x)\, dtdx \quad &\leqslant \quad \int_\alpha^\beta \int_0^T \varphi_x(t,x)^2 \zeta(t,x)\, dtdx, \\
&= \quad \int_\alpha^\beta \int_0^T (\varphi_t(t,x)^2 - a(x)\varphi(t,x)^2)\zeta(t,x) dtdx \\
&\quad\; + \int_\alpha^\beta \int_0^T ((\varphi\varphi_x)_x(t,x) - (\varphi\varphi_t)_t(t,x))\zeta(t,x) dtdx.
\end{aligned}$$

We now estimate the right hand side in this inequality. Integrating by parts, one gets

$$-\int_0^T \int_\alpha^\beta (\varphi\varphi_t)_t(t,x)\zeta(t,x)\, dxdt \leqslant \frac{\|\zeta_t\|_\infty}{2} \int_0^T \int_\alpha^\beta (\varphi_t^2(t,x) + \varphi^2(t,x))\, dxdt$$

and

$$\int_0^T \int_\alpha^\beta (\varphi\varphi_x)_x(t,x)\zeta(t,x) dxdt \leqslant \frac{\|\zeta_{xx}\|_\infty}{2} \int_0^T \int_\alpha^\beta \varphi^2(t,x) dxdt.$$

We then infer

$$\int_{\alpha+\eta}^{\beta-\eta} \int_{\eta}^{T-\eta} \varphi_x(t,x)^2\, dtdx \leqslant c(\eta) \int_0^T \int_\alpha^\beta \left(\varphi_t(t,x)^2 + \varphi(t,x)^2\right) dxdt,$$

where $c(\eta) = \frac{\|\zeta_t\|_\infty}{2} + \max\left\{\|\zeta\|_\infty, \frac{\|\zeta_{xx}\|_\infty}{2}\right\}$, for every $\eta \in (0, \beta - \alpha)$. The energy estimate

$$\int_{\eta}^{T-\eta} \int_{\alpha+\eta}^{\beta-\eta} \hat{E}_{k^2}(t,x)\, dxdt \leqslant (c(\eta) + \max\{1, \|a\|_\infty\}) \int_0^T \int_\alpha^\beta \left(\varphi_t(t,x)^2 + \varphi(t,x)^2\right)\, dxdt$$

is thus a consequence of the previous inequalities. Finally, choosing a particular function ζ enjoying the symmetry properties

$$\zeta(T-t,x) = \zeta(t,x), \quad \zeta(t,\alpha+y) = \zeta(t,\beta-y)$$

for every $t \in [0,T]$ and $y \in [0, \beta-\alpha]$, and defined by $\zeta(t,x) = \frac{t}{\eta}\frac{(x-\alpha)^2}{\eta^2}$ on $[0,\eta] \times [\alpha, \alpha+\eta]$ leads to the expected result. ☐

Remark 11. We stress that the particular choice of function ζ used above is motivated by the facts that the function $f : t \mapsto t/\eta$ solves the problem

$$\inf\{\|f'\|_\infty \mid f \in W^{1,\infty}(0,\eta),\ f(0)=0 \text{ and } f(\eta)=1\}$$

and the function $g : x \mapsto x^2/\eta^2$ solves the problem

$$\inf\{\|g''\|_\infty \mid g \in W^{2,\infty}(0,\eta),\ g(0)=0 \text{ and } g(\eta)=1\}.$$

With the notations of Figure 2.2, propagations properties of the wave equation are used in the following lemma to derive an energy estimate from Q_1 to P_1 and from Q_2 to P_2.

Lemma 7. *Let $T > T_0$ with $T_0 = 2\max\{\alpha, \pi-\beta\}$ and $\eta \in (0, \frac{\beta-\alpha}{4})$. Recall that $C(\eta)$ is the constant defined in the statement of Lemma 6.*

i) Introduce the positive number δ such that $\pi - \beta = \frac{T}{2} - 2\delta$. Then, one has

$$\int_{\frac{T}{2}-\delta}^{\frac{T}{2}+\delta}\int_{\beta-\eta}^{\pi} \hat{E}_{k^2}(t,x)\,dxdt \leqslant \frac{C(\eta)}{2k\eta}\left(e^{2k(\pi-\beta+3\eta)} - e^{4k\eta}\right)\int_0^T\int_\alpha^\beta (\varphi_t^2(t,\sigma)+\varphi^2(t,\sigma))\,d\sigma dt, \tag{2.27}$$

for every $\eta \in (0, \min\{\frac{\beta-\alpha}{4}, \frac{\delta}{4}\})$.

ii) Introduce the positive number δ' such that $\alpha = \frac{T}{2} - 2\delta'$. Then, one has

$$\int_{\frac{T}{2}-\delta'}^{\frac{T}{2}+\delta'}\int_{0}^{\alpha+\eta} \hat{E}_{k^2}(t,x)dxdt \leqslant \frac{C(\eta)}{2k\eta}(e^{-2k\eta} - e^{-2k(\alpha+2\eta)})\int_0^T\int_\alpha^\beta (\varphi_t^2(t,\sigma)+\varphi^2(t,\sigma))\,d\sigma dt, \tag{2.28}$$

for every $\eta \in (0, \min\{\frac{\beta-\alpha}{4}, \frac{\delta'}{4}\})$.

Démonstration. Let $\xi \in (\alpha, \beta-\eta)$ and $x \in (\xi, \min\{T+\xi, \pi\})$. Using an integration by parts [4], one gets

$$\begin{aligned}\frac{d}{dx}\int_{x-\xi}^{T-x+\xi} \hat{E}_{k^2}(t,x)dt &= -(\varphi_t-\varphi_x)^2(T+\xi-x,x) - (\varphi_t+\varphi_x)^2(x-\xi,x)\\ &\quad -k^2\left(\varphi^2(T+\xi-x,x)+\varphi^2(x-\xi,x)\right) + 4\int_{x-\xi}^{T-(x-\xi)} k^2\varphi_x\varphi\,dt,\\ &\leqslant 4\int_{x-\xi}^{T-(x-\xi)} k^2|\varphi_x\varphi|\,dt.\end{aligned}$$

We thus infer that

$$\frac{d}{dx}\int_{x-\xi}^{T-x+\xi} \hat{E}_{k^2}(t,x)dt \leqslant 2k\int_{x-\xi}^{T-x+\xi} \hat{E}_{k^2}(t,x)dt. \tag{2.29}$$

By noting that the function $x \in (\xi,\pi) \mapsto e^{-2k(x-\xi-\eta)}\int_{x-\xi}^{T-x+\xi} \hat{E}_{k^2}(t,x)dt$ is non increasing, it follows that

$$\int_{\frac{T}{2}-\delta}^{\frac{T}{2}+\delta} \hat{E}_{k^2}(t,x)\,dt \leqslant \int_{x-\xi}^{T-x+\xi} \hat{E}_{k^2}(t,x)\,dt \leqslant e^{2k(x-\xi-\eta)}\int_\eta^{T-\eta} \hat{E}_{k^2}(t,\xi+\eta)\,dt,$$

4. We also use that, for any function f regular enough, the following identity

$$\frac{d}{dx}\int_{x-\xi}^{T-x+\xi} f(t,x)dt = -f(T+\xi-x,x) - f(x-\xi,x) + \int_{x-\xi}^{T-x+\xi}\frac{\partial f}{\partial x}(t,x)dt$$

holds for every $x \in (\xi, \min\{T+\xi,\pi\})$.

for every $\eta \in (0, \frac{\delta}{4})$, $\xi \in [\beta-4\eta, \beta-2\eta]$ and $x \in [\beta-\eta, \pi]$. Integrating this inequality with respect to ξ on $[\beta-4\eta, \beta-3\eta]$ yields

$$\int_{\frac{T}{2}-\delta}^{\frac{T}{2}+\delta} \hat{E}_{k^2}(t,x)\,dt \leqslant \frac{e^{2k(x-\beta+3\eta)}}{\eta} \int_{\eta}^{T-\eta} \int_{\beta-3\eta}^{\beta-2\eta} \hat{E}_{k^2}(t,\sigma)\,dtd\sigma, \tag{2.30}$$

for every $x \in [\beta-\eta, \pi]$. Notice that $(\beta-3\eta, \beta-2\eta) \subset (\alpha+\eta, \beta-\eta)$ since $\eta \leqslant \frac{\beta-\alpha}{4}$. Hence, one deduces the expected result by combining the last inequality with the conclusion of Lemma 6.

It remains now to prove the second energy estimate. It suffices to mimic the first part of the proof, noting first that

$$\frac{d}{dx} \int_{\xi-x}^{T-\xi+x} \hat{E}_{k^2}(t,x)\,dt \geqslant 2k \int_{\xi-x}^{T-\xi+x} \hat{E}_{k^2}(t,x)\,dt, \tag{2.31}$$

for all $\xi \in (\alpha+\eta, \beta)$ and $x \in [0, \xi]$, and second that the function $x \in (0,\xi) \mapsto e^{-2k(x-\xi+\eta)} \int_{\xi-x}^{T-\xi+x} \hat{E}_{k^2}(t,x)\,dt$ is non decreasing. □

We now prove Theorem 2. With the notations of Lemma 7, introduce $\delta^- = \min\{\delta, \delta'\} = \frac{T-T_0}{4}$. Combining (2.26), (2.27) et (2.28) yields

$$\int_{\frac{T}{2}-\delta^-}^{\frac{T}{2}+\delta^-} \int_0^{\pi} \hat{E}_{k^2}(t,x)\,dxdt \leqslant K(\eta,\alpha,\beta,k) \int_0^T \int_{\alpha}^{\beta} (\varphi_t^2+\varphi^2)(t,x)\,dxdt, \tag{2.32}$$

where $K(\eta,\alpha,\beta,k)$ is defined by (2.11). According to (2.2), we infer

$$2\delta^- \left\| \varphi_t\left(\frac{T}{2},\cdot\right) \right\|^2_{L^2(0,\pi)} \leqslant 2\delta^- \int_0^{\pi} \hat{E}_a\left(\frac{T}{2},x\right)\,dx \leqslant K(\eta,\alpha,\beta,k) \int_0^T \int_{\alpha}^{\beta} (\varphi_t^2(t,x)+\varphi^2(t,x))\,dxdt, \tag{2.33}$$

for every $\eta \in (0, \min\{\frac{\delta^-}{4}, \frac{\beta-\alpha}{4}\})$ and $T > T_0$.

It remains now to compare the right hand in (2.33) with the observation term. Let $\gamma > 0$ such that $\gamma \leqslant \delta^-$ and let $\tau \in (\frac{T}{2}-\gamma, \frac{T}{2})$. Introduce the function ψ defined by

$$\begin{array}{rlcl} \psi: & (0,2\tau)\times(0,\pi) & \to & \mathbb{R} \\ & (t,x) & \mapsto & \int_{\tau}^{t} (\varphi_t(s,x)+\varphi_t(2\tau-s,x))\,ds. \end{array}$$

Notice that ψ satisfies the main equation of (2.1) on $(0,2\tau)\times\Omega$ and that $\psi_t(\tau,\cdot) = 2\varphi_t(\tau,\cdot)$. Furthermore, the function ψ clearly satisfies the inequality (2.33). Since $\tau > T_0/2$, we claim

$$\|\psi_t(\tau,\cdot)\|^2_{L^2(0,\pi)} \leqslant \frac{K(\eta,\alpha,\beta,k)}{2\mu^-} \int_0^{2\tau} \int_{\alpha}^{\beta} (\psi_t^2+\psi^2)(t,x)\,dxdt,$$

for every $\eta \in (0, \min\{\frac{\mu^-}{4}, \frac{\beta-\alpha}{4}\})$, where $\mu^- = \min\{\mu, \mu'\}$ with $\mu = \frac{\tau-\pi+\beta}{2}$, and $\mu' = \frac{\tau-\alpha}{2}$. We thus infer that

$$\begin{aligned} \frac{8\mu^-}{K(\eta,\alpha,\beta,k)} \|\varphi_t(\tau,\cdot)\|^2_{L^2(0,\pi)} \leqslant & \int_0^{2\tau} \int_{\alpha}^{\beta} (\varphi_t(t,x)+\varphi_t(2\tau-t,x))^2\,dxdt \\ & + \int_0^{2\tau} \int_{\alpha}^{\beta} \left(\int_{\tau}^{t} \varphi_t(s,x)+\varphi_t(2\tau-s,x)\,ds \right)^2 dxdt. \end{aligned}$$

Introducing

$$R_\tau = \int_0^{2\tau} \left(\int_\tau^t (\varphi_t(s,x) + \varphi_t(2\tau - s, x)) ds \right)^2 dt,$$

one gets using the Cauchy-Schwarz inequality

$$\begin{aligned} \frac{R_\tau}{2} &\leqslant \int_0^\tau (\tau - t) \int_t^\tau \varphi_t^2(s,x)\, dsdt + \int_\tau^{2\tau} (t-\tau) \int_\tau^t \varphi_t^2(s,x)\, dsdt, \\ &\quad + \int_0^\tau (\tau - t) \int_t^\tau \varphi_t^2(2\tau - s, x)\, dsdt + \int_\tau^{2\tau} (t-\tau) \int_\tau^t \varphi_t^2(2\tau - s, x)\, dsdt, \\ &\leqslant \tau^2 \int_0^{2\tau} \varphi_t^2(s,x)\, ds, \end{aligned}$$

and eventually

$$\|\varphi_t(\tau,\cdot)\|^2_{L^2(0,\pi)} \leqslant \frac{K(\eta,\alpha,\beta,k)}{2\mu^-} \left(1 + \frac{\tau^2}{2}\right) \int_0^{2\tau} \int_\alpha^\beta \varphi_t^2(t,x)\, dxdt.$$

Let us now provide an estimate of the right-hand side in the last inequality that is uniform with respect to τ. Using that $\tau \in (\frac{T}{2} - \gamma, \frac{T}{2})$,

$$\mu \geqslant \frac{\frac{T}{2} - \gamma - c + \beta}{2}, \quad \mu' \geqslant \frac{\frac{T}{2} - \gamma - \alpha}{2} \quad \text{and} \quad \mu^- \geqslant \frac{T - T_0}{4} - \frac{\gamma}{2}$$

and that the expression of $K(\eta,\alpha,\beta,k)$ above makes sense for $\eta \in (0, \min\{\frac{T-T_0}{16} - \frac{\gamma}{8}, \frac{\beta-\alpha}{4}\})$, we get

$$\int_{\frac{T}{2}-\gamma}^{\frac{T}{2}} \|\varphi_t(\tau,\cdot)\|^2_{L^2(0,\pi)} d\tau \leqslant \frac{2\gamma K(\eta,\alpha,\beta,k)}{T - T_0 - 2\gamma} \left(1 + \frac{T^2}{8}\right) \int_0^T \int_\alpha^\beta \varphi_t^2(t,x)\, dxdt.$$

Applying Lemma 5 leads to

$$\left(\frac{\gamma}{2} - \sup_{j\in\mathbf{N}^*} \frac{|\sin(\lambda_j\gamma)|}{2\lambda_j}\right) \frac{4(T - T_0 - 2\gamma)}{(8 + T^2)\gamma K(\eta,\alpha,\beta,k)} \leqslant \frac{\int_0^T \int_\alpha^\beta \varphi_t^2(t,x)\, dxdt}{\int_0^\pi \hat{E}_a(0,x)\, dx}, \tag{2.34}$$

providing hence a lower estimate of the observability constant.

For large values of T, it is yet possible to improve this estimate. Assuming that $T \geqslant 2T_0$, we write

$$\int_0^T \int_\alpha^\beta \varphi_t(t,x)^2\, dxdt \geqslant \int_0^{2T_0\left[\frac{T}{2T_0}\right]} \int_\alpha^\beta \varphi_t(t,x)^2\, dxdt = \sum_{i=0}^{\left[\frac{T}{2T_0}\right]-1} \int_{2iT_0}^{2(i+1)T_0} \int_\alpha^\beta \varphi_t(t,x)^2\, dxdt$$

and using the time reversibility of the wave equation, the inequality (2.34) improves into

$$\left[\frac{T}{2T_0}\right] \left(\frac{\gamma}{2} - \sup_{j\in\mathbf{N}^*} \frac{|\sin(\lambda_j\gamma)|}{2\lambda_j}\right) \frac{(T_0 - 2\gamma)}{(2 + T_0^2)\gamma K(\eta,\alpha,\beta,k)} \leqslant \frac{\int_0^T \int_\alpha^\beta \varphi_t^2(t,x)\, dxdt}{\int_0^\pi \hat{E}_a(0,x)\, dx}. \tag{2.35}$$

2.2.3 Proof of Corollary 1

Notice that, in the proof of Theorem 2, we only made local reasonings around the observation open set ω, and we never used the Dirichlet boundary conditions. Considering φ, the unique solution of the wave equation (2.1) with initial data $(\varphi(0,\cdot),\partial_t\varphi(0,\cdot))=(\varphi_0,\varphi_1)\in H_0^1(0,\pi)\times L^2(0,\pi)$. Let us apply the estimate of the observability constant proved for one open interval in Section 2.2.2 on $(0,T)\times(\beta_{i-1},\alpha_{i+1})$, we obtain using the notations of Theorem 2 and replacing (2.11) by (2.14)

$$C'_{T,T_0,a,\lambda}(\alpha_i,\beta_i)\int_{\beta_{i-1}}^{\alpha_{i+1}}\hat{E}_a(0,x)\,dx\leqslant\int_0^T\int_{\alpha_i}^{\beta_i}\partial_t\varphi(t,x)^2\,dxdt,$$

for every $i=1,\cdots,n$. Since $\int_0^\pi\hat{E}_a(0,x)\,dx\leqslant\sum_{i=1}^n\int_{\beta_{i-1}}^{\alpha_{i+1}}\hat{E}_a(0,x)\,dx$, it follows that

$$\min_{1\leqslant i\leqslant n}C'_{T,T_0,a,\lambda}(\alpha_i,\beta_i)\int_0^\pi\hat{E}_a(0,x)\,dx\leqslant\int_0^T\int_\omega\partial_t\varphi(t,x)^2\,dxdt,$$

whence the conclusion.

2.2.4 Proof of Corollary 2

Assume in a first time that $\omega=(\alpha,\beta)$. We follow the same lines as in the proof of Theorem 2, and propose a different way to conclude from the inequality (2.32). For this reason, we only provide here some explanations to adapt the proof of Theorem 2 to this simpler case.

Let us decompose the solution φ of (2.1) in the spectral basis $\{e_{a,j}\}_{j\in\mathbb{N}^*}$ as in (2.5)-(2.6). Since $j_0=\left[\frac{\|a\|_\infty-1}{2}\right]+1$, one gets by adapting the proof of Lemma 2,

$$\lambda_{j+1}-\lambda_j\geqslant\frac{1+2j_0-\|a\|_\infty}{j_0+1+\sqrt{j_0^2+\|a\|_\infty}}>0,\quad\text{for every }j\geqslant j_0.$$

Thus, $\lambda_{j+1}-\lambda_j\geqslant\gamma_{j_0}$ for every $j\in\mathbb{N}^*$.

Thus, applying Ingham's inequalities (see [53] and Remark 5) leads to

$$\begin{aligned}\int_0^T\int_\omega\varphi(t,x)^2\,dxdt&=\frac{1}{2}\int_0^T\int_\omega\left|\sum_{k\in\mathbb{Z}^*}\sqrt{a_{|k|}^2+b_{|k|}^2}e^{i\,\mathrm{sgn}(k)(\lambda_{|k|}t-\theta_{|k|})}e_{a,|k|}(x)\right|^2dxdt,\\&\leqslant\frac{5T}{\pi}\sum_{j=1}^{+\infty}(a_j^2+b_j^2)\int_\omega e_{a,j}(x)^2\,dx\end{aligned}$$

and

$$\begin{aligned}\int_0^T\int_\omega|\partial_t\varphi(t,x)|^2\,dxdt&=\frac{1}{2}\int_0^T\int_\omega\left|\sum_{k\in\mathbb{Z}^*}i\,\mathrm{sgn}(k)\lambda_{|k|}\sqrt{a_{|k|}^2+b_{|k|}^2}e^{i\,\mathrm{sgn}(k)(\lambda_{|k|}t-\theta_{|k|})}e_{a,|k|}(x)\right|^2dxdt,\\&\geqslant\frac{1}{\pi}\left(T-\frac{4\pi^2}{\gamma_{j_0}^2T}\right)\sum_{j=1}^{+\infty}\lambda_j^2(a_j^2+b_j^2)\int_\omega e_{a,j}(x)^2\,dx,\end{aligned}$$

for every $T > T_0'$. Using (2.19), one gets

$$\int_0^T \int_\omega \varphi(t,x)^2 \, dxdt \leqslant \frac{5\gamma_{j_0}^2 T^2}{(\gamma_{j_0}^2 T^2 - 4\pi^2)} \int_0^T \int_\omega |\partial_t \varphi(t,x)|^2 \, dxdt.$$

Combining this inequality with (2.33) yields

$$\frac{(T-T_0')(\gamma_{j_0}^2 T^2 - 4\pi^2)}{K(\eta,\alpha,\beta,\|a\|_\infty^{1/2})\left(2(\gamma_{j_0}^2 T^2 - 4\pi^2) + 10\gamma_{j_0}^2 T^2\right)} \int_0^\pi \hat{E}_a(0,x)\, dx$$
$$\leqslant \int_0^T \int_\alpha^\beta \partial_t \varphi(t,x)^2 \, dxdt,$$

for every $\eta \in (0, \min\{\frac{T-T_0}{16}, \frac{\beta-\alpha}{4}\})$, providing an estimate similar to (2.34) in this case. The conclusion follows, adapting the proof of Corollary 1 to extend the results to observation domains ω that are the finite union of open intervals.

2.3 Applications

2.3.1 Extension of the previous results to general wave equations

Let ℓ and T denote two positive constants. We consider the general one dimensional wave equation with Dirichlet boundary conditions

$$\begin{array}{ll} \partial_{tt}\phi(t,z) - \partial_z(b(z)\partial_z\phi)(t,z) + \overline{a}(z)\phi(t,z) = 0 & (t,z) \in (0,T)\times(0,\ell), \\ \phi(t,0) = \phi(t,\ell) = 0 & t \in [0,T], \\ \phi(0,z) = \phi_0(z),\ \partial_t\phi(0,z) = \phi_1(z) & z \in [0,\ell], \end{array} \tag{2.36}$$

where it is assumed that b and $\overline{a}$ denote nonnegative functions in $L^\infty(0,\ell)$ and that there exist a constant $b_0 > 0$ such that $b \geqslant b_0$ a.e. in $(0,\ell)$.

Recall that for every initial data $(\phi_0,\phi_1) \in H_0^1(0,\ell)\times L^2(0,\ell)$, Equation (2.36) has a unique solution $\phi \in C^0(0,T;H_0^1(0,\ell)) \cap C^1(0,T;L^2(0,\ell))$. Moreover, the equation (2.36) is said to be observable on a measurable subset $\omega \subset (0,\ell)$ in time T if there exists a positive constant c such that

$$\int_0^T \int_\omega \partial_t\phi(t,z)^2 \, dz\, dt \geqslant c \int_0^\ell \left(\phi_1(z)^2 + b(z)\phi_0'(z)^2 + \overline{a}(z)\phi_0(z)^2\right) dz.$$

Let us denote by $\overline{c}(T,\ell,b,\overline{a},\omega)$ the best constant in this inequality, that is

$$\overline{c}(T,\ell,b,\overline{a},\omega) = \inf_{\substack{(\phi_0,\phi_1)\in H_0^1(0,\ell)\times L^2(0,\ell)\\ (\phi_0,\phi_1)\neq(0,0)}} \frac{\int_0^T\int_\omega \partial_t\phi(t,z)^2\, dzdt}{\int_0^\ell \left(\phi_1(z)^2 + b(z)\phi_0'(z)^2 + \overline{a}(z)\phi_0(z)^2\right) dz}. \tag{2.37}$$

Notice in particular that, with the notations of Section 2.1, one has $c(T,a,\omega) = \overline{c}(T,\pi,\mathrm{Id},a,\omega)$.

The following result highlights the link between the observability of (2.36) and the observability of (2.1). For that purpose, let us introduce the standard change of variable for Sturm-Liouville equations

$$X : z \longmapsto \frac{\pi}{\ell'} \int_0^z \frac{ds}{\sqrt{b(s)}},$$

with $\ell' = \int_0^\ell \frac{ds}{\sqrt{b(s)}}$. Since $b \geqslant b_0$ a.e in $(0,\ell)$, the function X is continuous nondecreasing and defines thus a change of variable.

Proposition 1. *Let us assume that the function b belongs to $b \in W^{2,\infty}(0,\ell)$, let $\ell' = \int_0^\ell \frac{ds}{\sqrt{b(s)}}$ and $T' = \frac{\pi}{\ell'}T$. We define the function $a(\cdot)$ by*

$$a(x) = \frac{\ell'^2}{\pi^2}\left(\overline{a}(z) - \frac{1}{16}\frac{b'(z)^2}{b(z)} + \frac{1}{4}b''(z)\right), \tag{2.38}$$

for almost every $z \in (0,\ell)$ is nonnegative. Thus,

$$\overline{c}(T,\ell,b,\overline{a},\omega) = \frac{\ell'}{\pi}c(T',a,X(\omega)).$$

Démonstration. Using the change of variable $t' = \frac{\pi}{\ell'}t$ and introducing $\tilde{\phi}(t',\cdot) = \phi(t,\cdot)$, the main equation of (2.36) rewrites

$$\partial_{t't'}\tilde{\phi}(t',z) - \partial_z(\tilde{b}(z)\partial_z\tilde{\phi})(t,z) + \tilde{a}(z)\tilde{\phi}(t,z) = 0, \quad (t',z) \in (0,T') \times (0,\ell),$$

where $\tilde{b}(z) = \frac{\ell'^2}{\pi^2}b(z)$ and $\tilde{a}(z) = \frac{\ell'^2}{\pi^2}\overline{a}(z)$. Provided that $b \geqslant b_0$ and (2.38) are verified, ϕ is solution of (2.36) if and only if the function $\varphi : (t',x) \mapsto \tilde{b}(z)^{1/4}\tilde{\phi}(t',z)$ is solution of (2.1). The result follows, by writing that

$$\begin{aligned}\frac{\pi}{\ell'}\frac{\int_0^T\int_\omega \partial_t\phi(t,z)^2\,dzdt}{\int_0^\ell\left(\phi_1(z)^2 + b(z)\phi_0'(z)^2 + \overline{a}(z)\phi_0(z)^2\right)dz} &= \frac{\int_0^{T'}\int_\omega \partial_{t'}\tilde{\phi}(t',z)^2\,dzdt}{\int_0^\ell\left(\tilde{\phi}_{t'}(0,z)^2 + \tilde{b}(z)\tilde{\phi}_z(0,z)^2 + \tilde{a}(z)\tilde{\phi}(0,z)^2\right)dz},\\ &= \frac{\int_0^{T'}\int_{X(\omega)} \partial_{t'}\varphi(t',x)^2\,dxdt}{\int_0^\pi\left(\varphi_{t'}(0,x)^2 + \varphi_x(0,x)^2 + a(x)\varphi(0,x)^2\right)dx}.\end{aligned}$$

□

Using the correspondence between the observability of Equations (2.36) and (2.1), it is easy to deduce estimates of $\overline{c}(T,\ell,b,\overline{a},\omega)$ from the observability constants estimates in Theorems 1, 2, Corollaries 1 and 2, provided that the assumptions above be verified.

2.3.2 Evaluating the cost of control for the Hilbert Uniqueness Method.

Consider the internally controlled wave equation on $(0,\pi)$ with Dirichlet boundary conditions

$$\begin{array}{ll}\partial_{tt}y(t,x) - \partial_{xx}y(t,x) + a(x)y(t,x) = u(t,x) & (t,x) \in (0,T) \times (0,\pi),\\ y(t,0) = y(t,\pi) = 0 & t \in [0,T],\\ y(0,x) = y^0(x),\ \partial_t y(0,x) = y^1(x) & x \in (0,\pi),\end{array} \tag{2.39}$$

where u is a control supported in $[0,T] \times \omega$ where $\omega \subset (0,\pi)$ is Lebesgue measurable. Recall that for all initial data $(y^0,y^1) \in H_0^1(0,\pi) \times L^2(0,\pi)$ and every $u \in L^2((0,T) \times (0,\pi))$, the problem (2.39) is well posed and its solution y belongs to $C^0(0,T;H_0^1(0,\pi)) \cap C^1(0,T;L^2(0,\pi)) \cap C^2(0,T;H^{-1}(0,\pi))$.

In what follows, we will endow the space $H_0^1(0,\pi)$ with the inner product

$$(H_0^1(0,\pi))^2 \ni (\varphi,\psi) \longmapsto \int_0^\pi \varphi'(x)\psi'(x)\,dx + \int_0^\pi a(x)\varphi(x)\psi(x)\,dx,$$

instead of the standard inner-product of $H^1(0,\pi)$, according to the choice of norm in the energy space made to introduce the inequality (2.3). Therefore, the space $H^{-1}(0,\pi)$ will be endowed with the dual norm.

The exact null controllability problem settled in these spaces consists of finding a control u steering the control system (2.39) to $y(T,\cdot) = \partial_t y(T,\cdot) = 0$. It is well known that the Hilbert Uniqueness Method (HUM, see [73, 74]) provides a way to design the unique control solving the above exact null controllability problem and having moreover a minimal $L^2((0,T)\times(0,\pi))$ norm.

Using the observability inequality

$$\int_0^T \int_\omega \varphi(t,x)^2\,dxdt \geqslant c\|(\varphi^0,\varphi^1)\|^2_{L^2(0,\pi)\times H^{-1}(0,\pi)}, \tag{2.40}$$

where c is a positive constant (only depending on T and ω), valuable for every solution φ of the adjoint system

$$\begin{array}{ll} \partial_{tt}\varphi(t,x) - \partial_{xx}\varphi(t,x) + a(x)\varphi(t,x) = 0, & (t,x) \in (0,T)\times(0,\pi), \\ \varphi(t,0) = \varphi(t,\pi) = 0, & t \in [0,T], \\ \varphi(0,x) = \varphi^0(x),\ \partial_t\varphi(0,x) = \varphi^1(x), & x \in [0,\pi], \end{array} \tag{2.41}$$

and every $T \geqslant 2\pi$, the functional

$$J_\omega(\varphi^0,\varphi^1) = \frac{1}{2}\int_0^T \int_\omega \varphi(t,x)^2\,dxdt - \langle \varphi^1, y^0\rangle_{H^{-1},H_0^1} + \langle \varphi^0, y^1\rangle_{L^2}, \tag{2.42}$$

has a unique minimizer (still denoted (φ^0,φ^1)) in the space $L^2(0,\pi)\times H^{-1}(0,\pi)$, for all $(y^0,y^1) \in H_0^1(0,\pi)\times L^2(0,\pi)$. In (2.42) the notation $\langle\cdot,\cdot\rangle_{H^{-1},H_0^1}$ stands for the duality bracket between $H^{-1}(0,\pi)$ and $H_0^1(0,\pi)$, and the notation $\langle\cdot,\cdot\rangle_{L^2}$ stands for the usual scalar product of $L^2(0,\pi)$. The HUM control $u_{\min}$ steering (y^0,y^1) to $(0,0)$ in time T is then given by

$$u_{\min}(t,x) = \chi_\omega(x)\varphi(t,x), \tag{2.43}$$

for almost all $(t,x) \in (0,T)\times(0,\pi)$, where χ_ω denotes the characteristic function of the measurable set ω and φ is the solution of (2.41) with initial data (φ^0,φ^1) minimizing J_ω.

The HUM operator Γ_ω is then defined by

$$\begin{array}{rccc} \Gamma_\omega : & H_0^1(0,\pi)\times L^2(0,\pi) & \longrightarrow & L^2((0,T)\times(0,\pi)) \\ & (y^0,y^1) & \longmapsto & u_{\min} \end{array}$$

and the norm of the operator Γ_ω is given by

$$\|\Gamma_\omega\| = \sup\left\{ \frac{\|u_{\min}\|_{L^2((0,T)\times(0,\pi))}}{\|(y^0,y^1)\|_{H_0^1(0,\pi)\times L^2(0,\pi)}} \mid (y^0,y^1) \in H_0^1(0,\pi)\times L^2(0,\pi)\setminus\{(0,0)\} \right\}.$$

Recall that, by duality, the best constant in (2.40) coincides with the constant $c(T,a,\omega)$ defined by (2.4), that is the optimal constant in the observability inequality (2.3).

Proposition 2. *Let $T>0$ and let ω be measurable subset of $(0,\pi)$. If $c(T,a,\omega)>0$ then*

$$\|\Gamma_\omega\|^2 = \frac{1}{c(T,a,\omega)},$$

and if $c(T,a,\omega)=0$, then $\|\Gamma_\omega\|=+\infty$.

For a proof of this result, we refer for instance to [34, 86]. As a consequence, the results in Theorems 1, 2, Corollaries 1 and 2 permit to provide an estimate of the cost of control given by the Hilbert Uniqueness Method.

2.3.3 Estimating the stabilization rate of the damped wave equation.

Consider the damped wave equation on $(0,\pi)$ with Dirichlet boundary conditions

$$\begin{array}{ll} \partial_{tt}y(t,x)-\partial_{xx}y(t,x)+a(x)y(t,x)+2k\chi_\omega(x)\partial_t y(t,x)=0 & (t,x)\in(0,T)\times(0,\pi),\\ y(t,0)=y(t,\pi)=0 & t\in[0,T],\\ y(0,x)=y_0(x),\ \partial_t y(0,x)=y_1(x) & x\in(0,\pi), \end{array} \tag{2.44}$$

with $k>0$. Recall that for all initial data $(y_0,y_1)\in H^1_0(0,\pi)\times L^2(0,\pi)$, the problem (2.44) is well posed and its solution y belongs to $C^0(0,T;H^1_0(0,\pi))\cap C^1(0,T;L^2(0,\pi))$.

The energy associated to System (2.44) is defined by

$$E_{a,\omega}(t)=\int_0^\pi \left(\partial_t y(t,x)^2+\partial_x y(t,x)^2+a(x)y(t,x)^2\right)\,dx.$$

According to [20, 27], if ω has positive measure, this system is exponentially stable, i.e. its energy is known to obey

$$E_{a,\omega}(t)\leqslant E_{a,\omega}(0)e^{-\delta t} \tag{2.45}$$

for $t>0$, δ denotes a positive constant that does not depend on the initial data. We define the *decay rate* $\delta(a,\omega)$ as the largest such δ, in other words

$$\delta(a,\omega)=\sup\{\,\delta \mid E_{a,\omega}(t)\leqslant CE_{a,\omega}(0)e^{-\delta t},\text{ for every } t>0 \text{ and for every solution of (2.44)}\}.$$

According to the following proposition, one can provide an estimate of the constant δ (or $\delta(a,\omega)$) from the estimates proved in Theorems 1, 2, Corollaries 1 and 2.

Proposition 3. *Let ω be a measurable subset of $(0,\pi)$ and T_0 denotes the minimal observability time*[5]*. Thus, for every $(y_0,y_1)\in H^1_0(0,\pi)\times L^2(0,\pi)$ and $t\geqslant 2T_0$,*

$$E_{a,\omega}(t)\leqslant E_{a,\omega}(0)e^{-\delta t},$$

with

$$\delta=\frac{1}{2T_0}\ln\left(\frac{1+(1+T_0^2)c(T_0,a,\omega)}{(1+T_0^2)c(T_0,a,\omega)}\right),$$

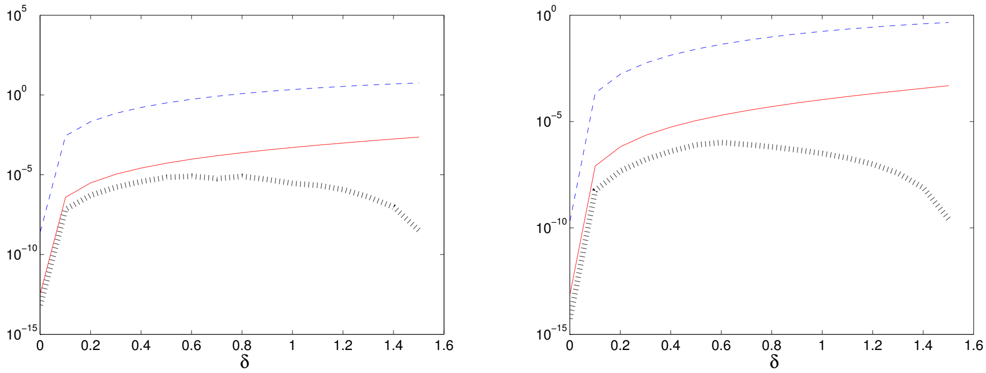
the constant $c(T_0,a,\omega)$ denoting the observability constant defined by (2.4).

For a proof of this result, we refer for instance to [44].

5. For instance, if $\omega=(\alpha,\beta)$ one has $T_0=2\max\{\alpha,\pi-\beta\}$

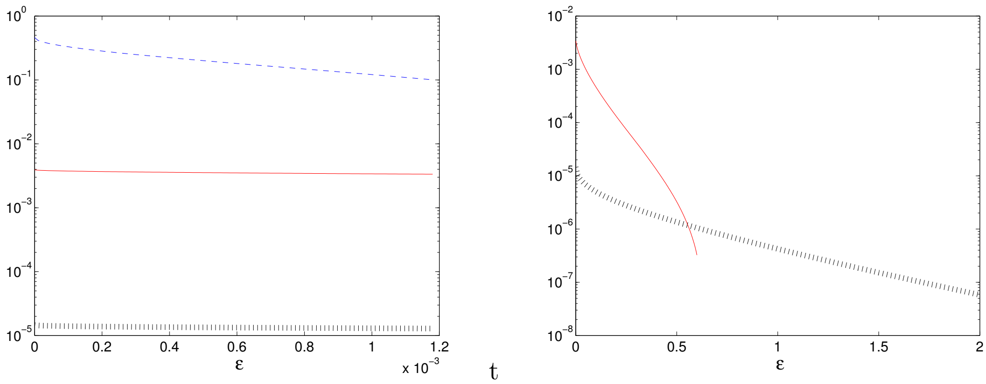
2.4 Examples and numerical illustrations

We provide here some numerical simulations and illustrations of observability constants estimates using both methods (spectral versus propagation).



(a) $T = 4\pi$ (b) Asymptotic behavior in T

FIGURE 2.3 – $a(.) = 0$ and $\omega = (0, \delta) \cup (\pi - \delta, \pi)$. Plots of $C_0(T, |\omega|)$ (- -), $C'_{T,T_0,0,\lambda}$ (//) and $\widetilde{C}_{T,T'_0,0,\lambda}$ (−) with respect to δ



(a) Small values of ε (b) Large values of ε

FIGURE 2.4 – $a : x \mapsto \varepsilon x$, $\omega = (0, \frac{3}{2})$ and $T = 8\pi$. Plots of $C_a(T, |\omega|)$ (- -), $C'_{T,T_0,a,\lambda}$ (//) and $\widetilde{C}_{T,T'_0,a,\lambda}$ (−) with respect to ε

The case $a(\cdot) = 0$ is investigated on Figure 2.3. We chose as observation subset $\omega = (0, \delta) \cup (\pi - \delta, \pi)$ with $0 \leqslant \delta \leqslant \pi$. In that case, the observability constant can be explicitly computed [6] whenever T is a multiple of 2π, as well as the limit of $c(T, a, \omega)/T$. To compute the estimates (2.12) and (2.13), we fully use Remark 9.

More precisely,

6. Let $\omega = (0, \delta) \cup (\pi - \delta, \pi)$ and T is a multiple of 2π. The observability constant is exactly

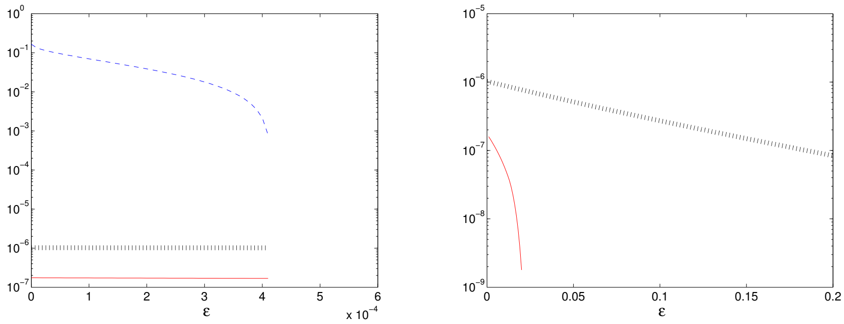
$$c(T, a, \omega) = C_0(T, |\omega|) = \left[\frac{T}{2\pi}\right] (2\delta - \sin(2\delta)),$$

— on Figure 2.3(a), the observation time is $T = 4\pi$. For such a choice of time T, the estimate $C_0(T, |\omega|)$ obtained by the spectral method coincides with the value of the observability constant. The graphs of the estimates obtained by each method, namely $C'_{T,T_0,0,\lambda}$ defined by (2.13) and $\widetilde{C}_{T,T_0',0,\lambda}$ defined by (2.15), are represented with respect to the parameter δ and compared with the observability constant $C_0(T, |\omega|)$.

— on Figure 2.3(b), the graphs of the limit of the estimates as T goes to $+\infty$ obtained by each method, namely $\frac{C'_{T,T_0,0,\lambda}}{T}$ and $\frac{\widetilde{C}_{T,T_0',0,\lambda}}{T}$, are represented with respect to the parameter δ and compared with the limit of the observability constant $\frac{C_0(T,|\omega|)}{T}$ as T goes to $+\infty$.

In accordance with Figure 2.3, one can easily show that the quantity $C'_{T,T_0,a,\lambda}$ decreases whenever the measure of ω is large enough, which is a confirmation that this estimate is not sharp.

On Figure 2.4, we consider the potential $a : x \mapsto \varepsilon x$ with $\varepsilon > 0$, the observation set $\omega = (0, \frac{3}{2})$ and the observation time $T = 8\pi$. More precisely, the graphs of the estimates obtained by each method, namely $C_a(T, |\omega|)$ defined by (2.8), $C'_{T,T_0,a,\lambda}$ defined by (2.13) and $\widetilde{C}_{T,T_0',a,\lambda}$ defined by (2.15), are represented with respect to the parameter ε. On Figure 2.4(b), only the graphs of $C'_{T,T_0,a,\lambda}$ (propagation method, Theorem 2) and $\widetilde{C}_{T,T_0',a,\lambda}$ (propagation method, Corollary 2) are plotted because of the smallness of the range of ε for which the spectral method makes sense (due to Condition (2.7)). Moreover, the range of ε for which $\widetilde{C}_{T,T_0',a,\lambda}$ is defined is restricted, since we use Ingham's inequalities (see the proof of Corollary 2 for more details). According to Figures 2.3 and 2.4(a), the spectral method seems more accurate than propagation methods, but needs the strong condition (2.7) on the potential function $a(\cdot)$ (see the comments and comparison between both methods at the end of the previous section). We provide on the figures 2.5, 2.6 and 2.7 several other examples illustrating each method.



(a) Small positive values of ε (b) Large positive values of ε

FIGURE 2.5 – $a : x \mapsto 2 + \varepsilon x^2$, $\omega = (0, \pi/3)$ and $T = 10\pi$. Plots of $C_0(T, |\omega|)$ (- -), $C'_{T,T_0,0,\lambda}$ (//) and $\widetilde{C}_{T,T_0',0,\lambda}$ (–) with respect to ε

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and one has the asymptotic expansion

$$c(T, a, \omega) \sim \frac{(2\delta - \sin(2\delta))}{2\pi} T.$$

as $T \to +\infty$.

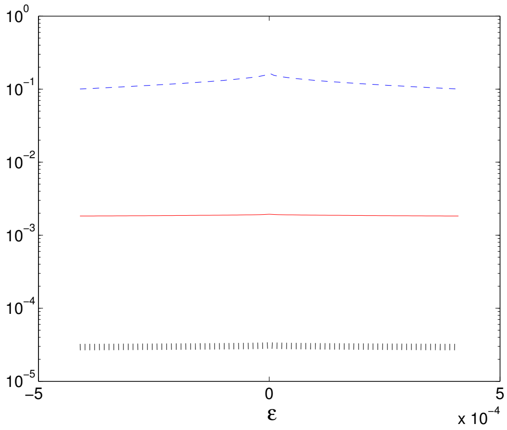

FIGURE 2.6 – $a : x \mapsto \varepsilon \cos x$, $\omega = (0, \frac{\pi}{3})$ and $T = 10\pi$. Plots of $C_a(T, |\omega|)$ (- -), $C'_{T,T_0,a,\lambda}$ (//) and $\widetilde{C}_{T,T'_0,a,\lambda}$ (−) with respect to ε

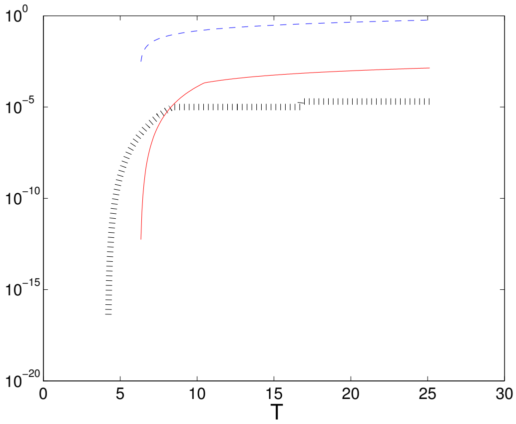

FIGURE 2.7 – $a : x \mapsto \frac{\beta}{1+4\pi\beta}\left(1 - \frac{\sin(|\omega|)}{|\omega|}\right)\frac{x}{\pi}$ with $\beta = \frac{(\ln 9 - \ln 8)^2}{2\pi^2}$ and $\omega = (0, \pi/3)$. Plots of $C_0(T, |\omega|)$ (- -), $C'_{T,T_0,0,\lambda}$ (//) and $\widetilde{C}_{T,T'_0,0,\lambda}$ (−) with respect to T

Chapitre 3

Non-localization of eigenfunctions for Sturm-Liouville operators

Joint work with **Pierre Lissy** and **Yannick Privat**.

Abstract : In this article, we investigate a non-localization property of the eigenfunctions of Sturm-Liouville operators $A_a = -\partial_{xx} + a(\cdot)\,\mathrm{Id}$ with Dirichlet boundary conditions, where $a(\cdot)$ runs over the bounded nonnegative potential functions on the interval $(0, L)$ with $L > 0$. More precisely, we address the extremal spectral problem of minimizing the L^2-norm of a function $e(\cdot)$ on a measurable subset ω of $(0, L)$, where $e(\cdot)$ runs over all eigenfunctions of A_a, at the same time with respect to all subsets ω having a prescribed measure and all L^∞ potential functions $a(\cdot)$ having a prescribed essentially upper bound. We provide some existence and qualitative properties of the minimizers, as well as precise lower and upper estimates on the optimal value. Numerous consequences in control and stabilization theory are then highlighted, both theoretically and numerically.

Keywords : Sturm-Liouville operators, eigenfunctions, extremal problems, calculus of variations, control theory, wave equation.

3.1 Introduction and main results

3.1.1 Localization/Non-localization of Sturm-Liouville eigenfunctions

In a recent survey article concerning the Laplace operator ([40]), D. Grebenkov and B.T. Nguyen introduce, recall and gather many possible definitions of the notion of *localization of eigenfunctions*. In particular, they consider, in the section 7.7 of their article, the Dirichlet-Laplace operator Δ on a given bounded open set Ω of $\mathbb{R}^n$, a Hilbert basis of eigenfunctions $(e_j)_{j\in\mathbb{N}^*}$ in $L^2(\Omega)$ and use as a measure of localization of the eigenfunctions on a measurable subset $\omega \subset \Omega$ the following criterion

$$C_p(\omega) = \inf_{j\in\mathbb{N}^*} \frac{\|e_j\|_{L^p(\omega)}}{\|e_j\|_{L^p(\Omega)}},$$

where $p \geqslant 1$. For instance, evaluating this quantity for different choices of subdomains ω if Ω is a ball or an ellipse allows to illustrate the so-called *whispering galleries* or *bouncing ball* phenomena. At the opposite, when Ω denotes the d-dimensional box $(0,\ell_1)\times\cdots\times(0,\ell_d)$ (with $\ell_1,\dots,\ell_d>0$), it is recalled that $C_p(\omega)>0$ for any $p\geqslant 1$ and any measurable subset $\omega\subset\Omega$ whenever the ratios $(\ell_i/\ell_j)^2$ are not rational numbers for every $i\neq j$.

Many other notions of localization have been introduced in the literature. Regarding the Dirichlet/Neumann/Robin Laplacian eigenfunctions on a bounded open domain Ω of $\mathbb{R}^n$ and using a semi-classical analysis point of view, the notions of *quantum limit* or *entropy* have been widely investigated (see e.g. [13, 18, 22, 37, 47, 56]) and provide an account for possible strong concentrations of eigenfunctions. Notice that the properties of $C_p(\omega)$ are intimately related to the behavior of high-frequency eigenfunctions and especially to the set of quantum limits of the sequence of eigenfunctions considered. Identifying such limits is a great challenge in quantum physics ([22, 37, 95]) and constitute a key ingredient to highlight non-localization/localization properties of the sequence of eigenfunctions considered.

Given a nonzero integer p, the *non-localization property* of a sequence $(e_j)_{j\in\mathbb{N}^*}$ of eigenfunctions means that the real number $C_p(\omega)$ is positive for every measurable subset $\omega\subset\Omega$ of positive measure. Regarding the one-dimensional Dirichlet-Laplace operator on $\Omega=(0,\pi)$, it has been highlighted in the case where $p=2$ (for instance in [46, 63, 87]) that

$$\inf_{|\omega|=r\pi} C_2(\omega)=\inf_{|\omega|=r\pi}\inf_{j\in\mathbb{N}^*}\frac{2}{\pi}\int_\omega \sin(jx)^2\,dx>0,$$

for every $r\in(0,1)$.

Motivated by these considerations, the present work is devoted to studying similar issues in the case $p=2$, for a general family of one-dimensional Sturm-Liouville operators of the kind $A_a=-\partial_{xx}+a(\cdot)\,\mathrm{Id}$ with Dirichlet boundary conditions, where $a(\cdot)$ is a negative essentially bounded potential defined on the interval $(0,L)$. More precisely, we aim at providing lower quantitative estimates of the quantity $C_2(\omega)$, where $(e_{a,j})_{j\in\mathbb{N}^*}$ denotes now a sequence of eigenfunctions of A_a, in terms of the measure of ω and the essential supremum of $a(\cdot)$ by minimizing this criterion at the same time with respect to ω and $a(\cdot)$, over the class of subsets ω having a prescribed measure and over a well-chosen class of potentials $a(\cdot)$, relevant from the point of view of applications. Independently of its intrinsic interest, the choice "$p=2$" is justified by the fact that the quantity $C_2(\omega)$ plays a crucial role in many control, stabilization or inverse problems (see Section 3.3.1,[46],[64]).

Let us mention that using the Liouville transform (see [26, Page 292]), the result obtained for the Sturm-Liouville operator A_a can be easily transferred to the operator $-\partial_x(w(x)\partial_x)$ for some weight function w belonging to an appropriate class of functions. We refer to [46, Section 3.1] for additional explanations and numerical computations.

The article is organized as follows : in Section 3.1.2, the extremal problem we will investigate is introduced. The main results of this article are stated in Section 3.1.3 : a comprehensive analysis of the extremal problem is performed, reducing in some sense (that will be made precise in the sequel) this infinite-dimensional problem to a finite one. We moreover provide very simple lower and upper estimates of the optimal value. The whole section 3.2 is devoted to the proofs of the main and intermediate results. Finally, consequences and applications of our main results for observation and control theory and several numerical illustrations and investigations are gathered in Section 3.3.

3.1.2 The extremal problem

Let L be a positive real number and $a(\cdot)$ be an essentially nonnegative function belonging to $L^\infty(0,L)$. We consider the operator

$$A_a := -\partial_{xx} + a(\cdot),$$

defined on $\mathcal{D}(A_a) = H_0^1(0,L) \cap H^2(0,L)$. As a self-adjoint operator, A_a admits a Hilbert basis of $L^2(0,L)$ made of eigenfunctions denoted $e_{a,j} \in \mathcal{D}(A_a)$ and there exists a sequence of increasing positive real numbers $(\lambda_{a,j})_{j\in\mathbf{N}^*}$ such that $e_{a,j}$ solves the eigenvalue problem

$$\begin{cases} -e''_{a,j}(x) + a(x)e_{a,j}(x) = \lambda_{a,j}^2 e_{a,j}(x), \ x \in (0,L), \\ \qquad e_{a,j}(0) = e_{a,j}(L) = 0. \end{cases} \tag{3.1}$$

By definition, the normalization condition

$$\int_0^L e_{a,j}^2(x)\,dx = 1 \tag{3.2}$$

is satisfied and we will also impose in the sequel that $e'_{a,j}(0) > 0$, so that the function $e_{a,j}$ is uniquely defined.

With regards to the explanations of Section 3.1.1, we are interested in the non-localization property of the sequence of eigenfunctions $(e_{a,j})_{j\in\mathbf{N}^*}$. The quantity of interest, denoted $J(a,\omega)$, is defined by

$$J(a,\omega) = \inf_{j\in\mathbf{N}^*} \frac{\int_\omega e_{a,j}^2(x)\,dx}{\int_0^L e_{a,j}^2(x)\,dx} = \inf_{j\in\mathbf{N}^*} \int_\omega e_{a,j}^2(x)\,dx, \tag{3.3}$$

where ω denotes a measurable subset of $(0,L)$ of positive measure.

The real number $J(a,\omega)$ is the equivalent for the Sturm-Liouville operators A_a of the quantity $C_2(\omega)$ introduced in Section 3.1.1 for the one-dimensional Dirichlet-Laplace operator.

It is natural to assume the knowledge of *a priori* informations about the subset ω and the potential function $a(\cdot)$. Indeed, we will choose them in some classes that are small enough to make the minimization problems we will deal with non-trivial, but also large enough to provide "explicit" (at least numerically) values of the criterion for a large family of potential.

Hence, in the sequel, we will assume that :

— the measure (or at least an lower bound of the measure) of the subset ω is given. Indeed, we will use in the sequel that prescribing either the measure of ω or a lower bound drives to the same solution of the optimal design problem we will consider (see Section 3.2.4;

— the potential function $a(\cdot)$ is nonnegative and essentially bounded.

We believe that such conditions are physically relevant and also adapted to the context of control or inverse problems.

Fix $M > 0$, $r \in (0,1)$, α and β two real numbers such that $\alpha < \beta$. Let us introduce the class of admissible observation subsets

$$\Omega_r(\alpha,\beta) = \{\text{Lebesgue measurable subset } \omega \text{ of } (\alpha,\beta) \text{ such that } |\omega| = r(\beta-\alpha)\}, \tag{3.4}$$

as well as the class of admissible potentials

$$\mathcal{A}_M(\alpha,\beta) = \{a \in L^\infty(\alpha,\beta) \text{ such that } 0 \leqslant a(x) \leqslant M \text{ a.e. on } (\alpha,\beta)\}, \tag{3.5}$$

in which the potential function $a(\cdot)$ will be chosen in the sequel.

Let us now introduce the optimal design problem we will investigate.

Extremal spectral problem. *Let $M > 0$, $r \in (0,1)$ and $L > 0$ be fixed. We consider*

$$m(L,M,r) = \inf_{a\in\mathcal{A}_M(0,L)} \inf_{\omega\in\Omega_r(0,L)} J(a,\omega), \qquad (\mathcal{P}_{L,r,M})$$

where the functional J is defined by (3.3), $\Omega_r(0,L)$ and $\mathcal{A}_M(0,L)$ are respectively defined by (3.4) and (3.5).

In the sequel, we will call *minimizer of the problem* ($\mathcal{P}_{L,r,M}$) a triple $(a^*,\omega^*,j^*) \in \mathcal{A}_M(0,L)\times \Omega_r(0,L)\times \mathbb{N}^*$ (whenever it exists) such that

$$\inf_{a\in\mathcal{A}_M(0,L)} \inf_{\omega\in\Omega_r(0,L)} J(a,\omega) = \int_{\omega^*} e_{a^*,j^*}(x)^2\,dx.$$

Remark 12. Noting that for a given real number c, the operator $A_a + c\,\mathrm{Id}$ has the same eigenfunctions as A_a, we claim that all the results and conclusions of this article will also hold by replacing the class of potentials $\mathcal{A}_M(\alpha,\beta)$ by the larger class

$$\widetilde{\mathcal{A}}_M(\alpha,\beta) := \left\{ a\in L^\infty(\alpha,\beta) \text{ such that } \sup_{(\alpha,\beta)} a(\cdot) - \inf_{(\alpha,\beta)} a(\cdot) \leqslant M \right\}.$$

Indeed, for all $\rho \in \widetilde{\mathcal{A}}_M(0,L)$, set $a = \rho - \inf_{(0,L)}\rho$. Then, every eigenfunction $e_{a,j}$ solving the system

$$\begin{cases} -e''_{a,j}(x) + a(x)e_{a,j}(x) = \lambda^2_{a,j}e_{a,j}(x), \ x\in(0,L), \\ \qquad e_{a,j}(0) = e_{a,j}(L) = 0 \end{cases}$$

is also a solution of the system

$$\begin{cases} -e''_{a,j}(x) + \rho(x)e_{a,j}(x) = \mu^2_{a,j}e_{a,j}(x), \ x\in(0,L), \\ \qquad e_{a,j}(0) = e_{a,j}(L) = 0, \end{cases}$$

with $\mu^2_{a,j} = \lambda^2_{a,j} - \inf_{(0,L)}\rho$, whence the claim.

3.1.3 Main results and comments

Let us state the main results of this article. The next theorems are devoted to the analysis of the optimal design problems ($\mathcal{P}_{L,r,M}$).

We also stress on the fact that the estimates of $J(a,\omega)$ in the following result are valuable for every measurable subset ω of prescribed measure and that we do not need to make additional topological assumption on it.

Theorem 3. *Let $r \in (0,1)$ and $M \in \mathbb{R}_+^*$.*

i Problem ($\mathcal{P}_{L,r,M}$) has a solution (j_0,ω^*,a^*). In particular, there holds

$$m(L,M,r) \ = \ \min_{a\in\mathcal{A}_M(0,L)} \min_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j_0}(x)^2\,dx,$$

and the solution a^* of Problem ($\mathcal{P}_{L,r,M}$) is bang-bang, that is to say equal to 0 or M a.e. in $(0,L)$.

ii Assume that $M \in (0, \pi^2/L^2]$. Then, ω^* is the union of j_0+1 intervals, and a^* has at most $3j_0-1$ and at least j_0 switching points[1]*.*
Moreover, one has the estimate

$$\gamma \min(r, \underline{r}_2)^3 \leqslant m(L,M,r) \leqslant r - \frac{\sin(\pi r)}{\pi}, \tag{3.6}$$

with $\gamma = \frac{7\sqrt{3}}{8}(3-2\sqrt{2}) \simeq 0.2600$ and $\underline{r}_2 = \frac{\sqrt{5}}{5} + \frac{\sqrt{10}}{10} \simeq 0.7634$.

The estimate (3.6) is in fact inferred from a more precise estimate for the optimal design problem

$$m_j(L,M,r) = \inf_{a\in\mathcal{A}_M(0,L)} \inf_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j}(x)^2\,dx, \tag{$\mathcal{P}_{j,L,r,M}$}$$

with $j \in \mathbb{N}^*$. Because of its intrinsic interest, we state this estimate in the following proposition, which constitutes therefore an essential ingredient for the proof of the last point of Theorem 3.

Proposition 4. *Let $r \in (0,1)$ and let us assume that $M \in (0, \pi^2/L^2]$. Then, there holds*

$$m_j(L,M,r) \geqslant \min(r, \underline{r}_j)^3 \underline{m}_j, \tag{3.7}$$

for every $j \in \mathbb{N}^*$, where the sequences $(\underline{m}_j)_{j\in\mathbb{N}^*}$ and $(\underline{r}_j)_{j\in\mathbb{N}^*}$ are defined by

$$\underline{m}_j = \begin{cases} \frac{1}{2} & \text{if } j=1,\\ \dfrac{(2j^2-1)(j^2-1)^{\frac{3}{2}}\left(\sqrt{\frac{j^2}{j^2-2}}-1\right)^2}{3j^3\left(\left(\frac{j^2}{j^2-2}\right)^{\frac{j}{2}}-1\right)^2} & \text{if } j\geqslant 2,\end{cases}$$

and

$$\underline{r}_j = \begin{cases} 1 & \text{if } j=1,\\ \left(\dfrac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \dfrac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right) & \text{if } j\geqslant 2.\end{cases}$$

In the following result, we highlight the necessity of imposing a pointwise upper bound on the potentials functions $a(\cdot)$ to get the existence of a minimizer. Indeed, we have the following non-existence result when the pointwise constraint "$a(\cdot) \leqslant M$" is not imposed anymore.

Theorem 4. *Let $r \in (0,1)$ and $j \in \mathbb{N}^*$. Then, the optimal design problem of finding a minimizer to*

$$\inf_{a\in\mathcal{A}_\infty(0,L)} \inf_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j}(x)^2\,dx, \tag{$\mathcal{P}_{j,L,r,\infty}$}$$

where $\mathcal{A}_\infty(0,L) = \cup_{M>0}\mathcal{A}_M(0,L)$, has no solution.

We conclude this section by some remarks and comments on the main results we have just stated.

Remark 13. As we will see later, the proofs of point (i), the first part of point (ii) are quite classical and rests upon classical optimization tools. The main difficulty was to prove the left inequality of (3.6), which comes from the more refine estimates (3.7).

Remark 14. Let us highlight the interest of Theorem 3 for numerical investigations. Indeed, in view of providing numerical lower bounds of the quantity $J(a,\omega)$, Theorem 3 enables us to

1. Recall that a switching point of a *bang-bang* function is a point at which this function is not continuous.

reduce the solving of the infinite-dimensional problems $(\mathcal{P}_{L,r,M})$ to the solving of finite ones, since one has just to choose the optimal $3j_0^*-1$ switching points defining the best potential function a^* and to remark that necessarily ω^* is uniquely defined once a^* is defined (since it is defined in terms of a precise level set of e_{a^*,j_0^*}, see Proposition 5). We will strongly use this remark in Section 3.3.2.1, where illustrations and applications of Problem $(\mathcal{P}_{L,r,M})$ are developed.

Remark 15. As it will be emphasized in the sequel, the restriction on the range of values of the parameter M in the second point of Theorem 3 makes each eigenfunction $e_{a,j}$ convex or concave on each nodal domain. The upper bound on the parameter M, namely the real number π^2/L^2 correspond to the lowest eigenvalue of the Dirichlet Laplacian A_0. It constitutes a crucial element of the proofs of Theorem 3 and Proposition 4, that allows to compare each quantity $\int_\omega e_{a,j}(x)^2\,dx$ with the integral of the square of well-chosen piecewise affine functions. Unfortunately, the solving of Problem $(\mathcal{P}_{L,r,M})$ when $M > \pi^2/L^2$ appears much more intricate and cannot be handled with the same kinds of arguments.

Remark 16. According to Section 3.3.2.1, numerical simulations seem to indicate, at least in the case where $M = \frac{\pi^2}{L^2}$, that there exists a triple (j_0^*, ω^*, a^*) such that $j_0^* = 1$, the set ω^* and the graph of a^* are symmetric with respect to $L/2$ and a^* is a *bang-bang* function having exactly two switching points, solving Problem $(\mathcal{P}_{L,r,M})$. We were unfortunately unable to prove this assertion.

A first step would consist in finding an upper estimate of the optimal index j_0^*. Even this question appears difficult in particular since it is not obvious to compare the real numbers $m_j(L,M,r)$ for different indexes j. One of the reasons of that comes from the fact that the cost functional we considered does not write as the minimum of an energy function, making the comparison between eigenfunctions of different orders intricate.

Remark 17. It can be noticed that the lower bound in Proposition 4 is independent of the parameter L. This can be justified by using a rescaling argument allowing in particular to restrict our numerical investigations to the case where $L = \pi$ (for instance).

Lemma 8. *Let $j \in \mathbb{N}^*$, $r \in (0,1)$, $M > 0$ and $L > 0$. Then, there holds*

$$\inf_{a\in\mathcal{A}_M(0,\pi)}\ \inf_{\omega\in\Omega_r(0,\pi)} \int_\omega e_{a,j}(x)^2\,dx = \inf_{a\in\mathcal{A}_{\frac{M\pi^2}{L^2}}(0,L)}\ \inf_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j}(x)^2\,dx, \tag{3.8}$$

The proof of this lemma rests upon the simple change of variable $x = \pi y/L$.

Remark 18. The estimate (3.6) can be considered as sharp with respect to the parameter r, at least for r small enough (which is the most interesting case, at least in view of the applications). Indeed, there holds $\frac{\pi r-\sin(\pi r)}{\pi} \sim \frac{\pi^2}{6} r^3$ as r tends to 0, which shows that the power of r in the left-hand side cannot be improved. The graphs of the quantities appearing in the left and right-hand sides of (3.6) with respect to r are plotted on Figure 3.1 below.

An interesting issue would thus consist in determining the optimal bounds in the estimate (3.6), namely

$$\ell_- = \inf_{r\in(0,1)} \frac{m(L,M,r)}{r^3} \qquad \text{and} \qquad \ell_+ = \sup_{r\in(0,1)} \frac{m(L,M,r)}{r^3}. \tag{$\mathcal{P}_{\ell_-,\ell_+}$}$$

It is likely that investigating this issue would rely on a refined study of the optimality conditions of the problems $(\mathcal{P}_{\ell_-,\ell_+})$, but also of the problem $(\mathcal{P}_{L,r,M})$. According to (3.6), we know that $\ell_- \in [\gamma(\frac{\sqrt{5}}{5} + \frac{\sqrt{10}}{10})^3, 1]$ and $\ell_+ \in [\gamma, \pi^2/6]$.

Remark 19 (*Behavior of the sequence* $(\underline{m}_j)_{j\in\mathbb{N}^*}$ *and* $(\underline{r}_j)_{j\in\mathbb{N}^*}$*.*)**.** One can show by using tedious computations inspired by those of Appendix 3.4.3 that the sequences $(\underline{m}_j)_{j\in\mathbb{N}^*}$ and $(\underline{r}_j)_{j\in\mathbb{N}^*}$ are increasing. Moreover, straightforward computations show that $(\underline{m}_j)_{j\in\mathbb{N}^*}$ converges to $2/3$ and $(\underline{r}_j)_{j\in\mathbb{N}^*}$ converges to 1 as j tends to $+\infty$.

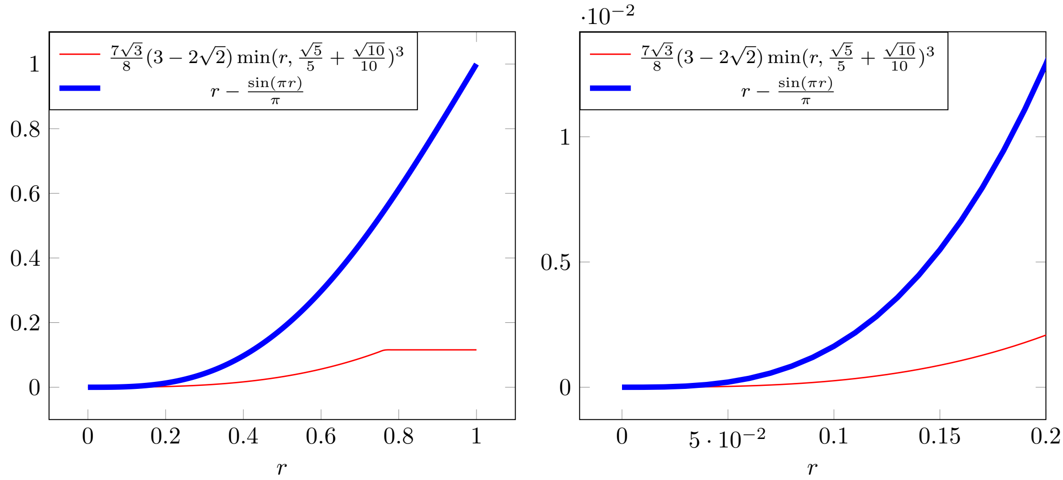

FIGURE 3.1 – (Left) Plots of $r \mapsto \gamma\min(r, \frac{\sqrt{5}}{5}+\frac{\sqrt{10}}{10})^3$ (thin line) and $x \mapsto (\pi r - \sin(\pi r))/\pi$ (bold line). (Right) Zoom on the graph for the range of values $r \in [0, 0.2]$.

3.2 Proofs of Theorem 3, Proposition 4, and Theorem 4

3.2.1 Preliminary material : existence results and optimality conditions

We gather in this section several results we will need in the sequel to prove Theorem 3, Proposition 4, and Theorem 4.

Let us first investigate the following simpler optimal design problem, where the potential $a(\cdot)$ is now assumed to be fixed, and which will constitute an important ingredient in the proof.

Auxiliary problem : fixed potential. *For a given* $j \in \mathbb{N}^*$, $M > 0$, $r \in (0,1)$ *and* $a \in \mathcal{A}_M(0,L)$, *we investigate the optimal design problem*

$$\inf_{\chi\in\mathcal{U}_r} \int_0^L \chi(x) e_{a,j}(x)^2\, dx, \tag{Aux-Pb}$$

where the set $\mathcal{U}_r$ *is defined by*

$$\mathcal{U}_r = \left\{ \chi \in L^\infty(0,L) \mid 0 \leqslant \chi \leqslant 1 \text{ a.e. in } (0,L) \text{ and } \int_0^L \chi(x)\, dx = rL \right\}. \tag{3.9}$$

We provide a characterization of the solutions of Problem (Aux-Pb).

Proposition 5. *Let* $r \in (0,1)$*. The optimal design problem Problem (Aux-Pb) has a unique solution that writes as the characteristic function of a measurable set* ω^* *of Lebesgue measure*

rL. Moreover, there exists a positive real number τ such that ω^* is a solution of Problem (Aux-Pb) if and only if

$$\omega^* = \{e_{a,j}(x)^2 < \tau\}, \tag{3.10}$$

up to a set of zero Lebesgue measure.

In other words, any optimal set, solution of Problem (Aux-Pb), is characterized in terms of the level set of the function $e_{a,j}(\cdot)^2$.

A proof of this result can be found in [84, Theorem 1] or [90, Chapter 1]. It yields the existence of a positive real number τ such that ω^* is a solution of Problem (Aux-Pb) if and only if

$$\chi_{\{e_{a,j}(x)^2<\tau\}} \leqslant \chi_{\omega^*}(x) \leqslant \chi_{\{e_{a,j}(x)^2\leqslant\tau\}}(x),$$

for almost every $x \in (0, L)$, by using a standard argument of decreasing rearrangement. We also mention [50, 59] as references on rearrangements and symmetrization of functions and their use in shape optimization. The conclusion follows by noting that for every $c > 0$, the set $\{e_{a,j}^2 = c\}$ has zero Lebesgue measure, by using standard properties of eigenfunctions of Sturm-Liouville operators.

Let us now investigate the continuity of the mapping $a \mapsto e_{a,j}$.

Lemma 9. *Let $M \in \mathbb{R}_+^*$ and $j \in \mathbb{N}^*$. Let us endow the space $\mathcal{A}_M(0, L)$ with the weak-$\star$ topology of $L^\infty(0, L)$ [2] and the space $H_0^1(0, L)$ with the standard strong topology inherited from the Sobolev norm $\|\cdot\|_{H_0^1}$. Then the function $a \in \mathcal{A}_M(0, L) \mapsto e_{a,j} \in H_0^1(0, L)$ is continuous.*

The principle of the proof of this lemma is standard, and is roughly recalled for the sake of completeness in Appendix 3.4.1. The following existence result is a direct consequence of this lemma.

Proposition 6. *Let $M \in \mathbb{R}_+^*$, $j \in \mathbb{N}^*$ and $r \in (0, 1)$. The optimal design Problem ($\mathcal{P}_{j,L,r,M}$) has at least one solution (a_j^*, ω_j^*).*

Proof of Proposition 6. Let us consider a relaxed version of the optimal design Problem ($\mathcal{P}_{j,L,r,M}$), where the characteristic function of ω has been replaced by a function χ in $\mathcal{U}_r$ where $\mathcal{U}_r$ is defined by (3.9). This relaxed version of ($\mathcal{P}_{j,L,r,M}$) writes

$$\inf_{(a,\chi)\in\mathcal{A}_M(0,L)\times\mathcal{U}_r} \int_0^L \chi(x)e_{a,j}(x)^2\,dx.$$

Let us endow $\mathcal{A}_M(0, L)$ and $\mathcal{U}_r$ with the weak-$\star$ topology of $L^\infty(0, L)$. Thus, both sets are compact. Moreover, according to Lemma 9 and since it is linear in the variable χ, the functional $(a, \chi) \in \mathcal{A}_M(0, L) \times \mathcal{U}_r \mapsto \int_0^L \chi(x)e_{a,j}(x)^2\,dx$ is continuous. The existence of a solution (a_j^*, χ_j^*) follows for the relaxed problem. Finally, noting that

$$\inf_{(a,\chi)\in\mathcal{A}_M(0,L)\times\mathcal{U}_r} \int_0^L \chi(x)e_{a,j}(x)^2\,dx = \inf_{\chi\in\mathcal{U}_r} \int_0^L \chi(x)e_{a_j^*,j}(x)^2\,dx,$$

there exists a measurable set ω_j^* of measure rL such that $\chi_j^* = \chi_{\omega_j^*}$, by Proposition 5.

The existence of a solution of Problem ($\mathcal{P}_{j,L,r,M}$) is then proved and there holds

$$\inf_{(a,\chi)\in\mathcal{A}_M(0,L)\times\mathcal{U}_r} \int_0^L \chi(x)e_{a,j}(x)^2\,dx = \inf_{a\in\mathcal{A}_M(0,L)} \int_{\omega^*} e_{a,j}(x)^2\,dx.$$

2. Recall that a sequence $(a_n)_{n\in\mathbb{N}^*}$ of $L^\infty(0, L)$ converges to a for the weak-$\star$ topology of $L^\infty(0, L)$ whenever $\int_0^L a_n(x)\varphi(x)$ dx converge to $\int_0^L a(x)\varphi(x)\,dx$ for every $\varphi \in L^1(0, L)$.

■

We now state necessary first order optimality conditions that enable us to characterize every critical point (a_j^*, ω_j^*) of the optimal design problem $(\mathcal{P}_{j,L,r,M})$.

Proposition 7. *Let $j \in \mathbb{N}^*$, $r \in (0,1)$ and $M > 0$. Let (a_j^*, ω_j^*) be a solution of the optimal design problem $(\mathcal{P}_{j,L,r,M})$ and let*

$$0 = x_j^0 < x_j^1 < x_j^2 < \cdots < x_j^{j-1} < L = x_j^j \tag{3.11}$$

be the $j+1$ zeros[3] *of the j-th eigenfunction $e_{a_j^*,j}$.*

For $i \in \{1 \dots j\}$, let us denote by $a_{j,i}^*$ the restriction of the function a_j^* to the interval (x_j^{i-1}, x_j^i) and by $\omega_{j,i}^*$ the set $\omega_j^* \cap (x_j^{i-1}, x_j^i)$. Then, necessarily, there exists $\tau \in \mathbb{R}_+^*$ such that

— *one has $\omega_j^* = \{e_{a_j^*,j}(x)^2 < \tau\}$ up to a set of zero Lebesgue measure,*

— *one has*

$$M\chi_{\{p_j(x)e_{a_j^*,j}(x)>0\}} \leqslant a_j^*(x) \leqslant M\chi_{\{p_j(x)e_{a_j^*,j}(x)\geqslant 0\}}(x), \tag{3.12}$$

for almost every $x \in (0,L)$, where p_j is defined piecewisely as follows : for $i \in \{1 \dots j\}$, the restriction of p_j to the interval (x_j^{i-1}, x_j^i) is denoted $p_{j,i}$, and $p_{j,i}$ is the (unique) solution of the adjoint system

$$\begin{cases} -p_{j,i}''(x) + (a_{j,i}^*(x) - \lambda_{a_j^*,j}^2)p_{j,i}(x) = (\chi_{\omega_{j,i}^*} - \tilde{c}_{j,i})e_{a_j^*,j}, & x \in (x_j^{i-1}, x_j^i), \\ p_{j,i}(x_j^{i-1}) = p_{j,i}(x_j^i) = 0, \end{cases} \tag{3.13}$$

verifying moreover

$$\int_{x_j^{i-1}}^{x_j^i} p_{j,i}(x)e_{a_{j,i}^*,j}(x)\,dx = 0, \tag{3.14}$$

where $\tilde{c_{j,i}}$ is given by

$$\tilde{c}_{j,i} = \frac{\int_{x_j^{i-1}}^{x_j^i} \chi_{\omega_{j,i}^*}(x)e_{a_j^*,j}^2(x)\,dx}{\int_{x_j^{i-1}}^{x_j^i} e_{a_j^*,j}^2(x)\,dx}. \tag{3.15}$$

In other words, any optimal set solution of Problem $(\mathcal{P}_{j,L,r,M})$ is characterized in terms of a level set of the function $e_{a_j^*,j}(\cdot)^2$ and the optimal potential function is characterized in terms of a level set of the function $p_j(\cdot)e_{a_j^*,j}(\cdot)$.

Remark 20. i Note that, according to Fredholm's alternative (see for example [49]), System (3.13)-(3.14) has a unique solution.

Before proving this proposition, let us state a technical lemma about the differentiability of the eigenfunctions $e_{a,j}$ with respect to a.

Lemma 10. *Let us endow the space $\mathcal{A}_M(0,L)$ with the weak-$\star$ topology of $L^\infty(0,L)$ and the space $H_0^1(0,L)$ with the standard strong topology inherited from the Sobolev norm $\|\cdot\|_{H^1}$. Let $a \in \mathcal{A}_M(0,L)$ and $\mathcal{T}_{a,\mathcal{A}_M(0,L)}$ the tangent cone*[4] *to the set $\mathcal{A}_M(0,L)$ at point a. For every $h \in$*

3. Recall that the family $\{x_j^k\}_{0\leqslant k\leqslant j}$ is the set of nodal points of the eigenfunction $e_{a^*,j}$ that are known to be simple.

4. That is the set of functions $h \in L^\infty(0,L)$ such that, for any sequence of positive real numbers ε_n decreasing to 0, there exists a sequence of functions $h_n \in L^\infty(0,L)$ converging to h as $n \to +\infty$, and $a + \varepsilon_n h_n \in \mathcal{A}_M(0,L)$ for every $n \in \mathbb{N}$ (see for instance [50, chapter 7]).

$\mathcal{T}_{a,\mathcal{A}_M(0,L)}$, *the mapping* $a \mapsto e_{a,j}$ *is Gâteaux-differentiable in the direction* h, *and its derivative, denoted* $\dot{e}_{a,j}$, *is the (unique) solution of*

$$\begin{cases} -\dot{e}''_{a,j}(x) + a(x)\dot{e}_{a,j}(x) + h(x)e_{a,j}(x) = \left(\dot{\lambda^2_{a,j}}\right)e_{a,j}(x) + \lambda^2_{a,j}\dot{e}_{a,j}(x), \ x \in (0,L), \\ \dot{e}_{a,j}(0) = \dot{e}_{a,j}(L) = 0, \end{cases} \tag{3.16}$$

with $\left(\dot{\lambda^2_{a,j}}\right) = \int_0^L h(x)e_{a,j}(x)^2\,dx$.

The proof of the differentiability is completely standard and is based on the fact that the eigenvalues $\lambda_{a,j}$ are simple. For this reason, we skip it and refer to [58, pages 375 and 425].

Proof of Proposition 7. The first point results from Proposition 5.

Let us show the second point. Let us now compute the Gâteaux-derivative of the cost functional $a \mapsto J_{\omega_j^*}(a)$, where

$$J_{\omega_j^*}(a) = J(a,\omega_j^*) = \int_{\omega_j^*} e_{a,j}(x)^2\,dx,$$

at $a = a_j^*$ in the direction $h_j \in \mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$. We denote it $\langle dJ_{\omega_j^*}(a_j^*), h\rangle$ and there holds

$$\langle dJ_{\omega_j^*}(a_j^*), h_j\rangle = 2\int_{\omega^*} \dot{e}_{a_j^*,j}(x)e_{a_j^*,j}(x)\,dx,$$

where $\dot{e}_{a_j^*,j}$ is the solution of (3.16).

Let us write this quantity in a more convenient form to state the necessary first order optimality conditions. Let h_j be an element of the tangent cone $\mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$. Let us write $h_j = \sum_{i=1}^{j-1} h_{j,i}$ where $h_{j,i} = h_j\chi_{(x_j^{i-1},x_j^i)}$ for all $i \in \{1\dots j\}$. Hence, $h_{j,i} \in \mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$ and there holds

$$\langle dJ_{\omega_j^*}(a_j^*), h_j\rangle = \sum_{i=1}^{j}\langle dJ_{\omega_j^*}(a_j^*), h_{j,i}\rangle.$$

It follows that it is enough to consider perturbations with compact support contained in each nodal domain in order to compute the Gâteaux-derivative of $J_{\omega_j^*}$. We will use for that purpose the adjoint state p_j piecewisely defined by (3.13)-(3.14).

Fix $i \in \{1\dots j\}$ and let $h_{j,i}$ be an element of the tangent cone $\mathcal{T}_{a_j^*,\mathcal{A}_M(x_j^{i-1},x_j^i)}$. Let us multiply the first line of (3.13) by $\dot{e}_{a_{j,i}^*,1}$ and then integrate by parts. We get

$$\int_{x_j^{i-1}}^{x_j^i} \dot{e}'_{a_{j,i}^*,1}(x)p'_{j,i}(x) + (a_{j,i}^*(x) - \lambda^2_{a_{j,i}^*,1})\dot{e}_{a_{j,i}^*,1}(x)p_{j,i}(x)\,dx = \frac{1}{2}\langle dJ(a_j^*), h_{j,i}\rangle. \tag{3.17}$$

Similarly, let us multiply the first line of (3.16) by $p_{j,i}$ and then integrate by parts. We get

$$\begin{aligned} &\int_{x_j^{i-1}}^{x_j^i} \dot{e}'_{a_{j,i}^*,1}(x)p'_{j,i}(x) + (a_{j,i}^*(x) - \lambda^2_{a_{j,i}^*,1})\dot{e}_{a_{j,i},1}(x)p_{j,i}(x)\,dx \\ &= \left(\dot{\lambda^2_{a_j^*,j}}\right)\int_{x_j^{i-1}}^{x_j^i} e_{a_{j,i},1}(x)p_{j,i}(x)\,dx - \int_{x_j^{i-1}}^{x_j^i} h_{j,i}(x)e_{a_{j,i},1}(x)p_{j,i}(x)\,dx. \end{aligned} \tag{3.18}$$

Combining (3.17) with (3.18) and using the condition (3.14) yields

$$\langle dJ_{\omega_j^*}(a_j^*), h_{j,i}\rangle = -2\int_{x_j^{i-1}}^{x_j^i} h_{j,i}(x)e_{a_{j,i}^*,1}(x)p_{j,i}(x)\,dx. \tag{3.19}$$

As a result, for a general $h_j \in \mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$, there holds

$$\langle dJ_{\omega_j^*}(a_j^*), h_j\rangle = -2\sum_{i=1}^{j}\int_{x_j^{i-1}}^{x_j^i} h_{j,i}(x)p_{j,i}(x)e_{a_{j,i}^*,1}\,dx = -2\int_0^L h_j(x)e_{a_j^*,j}(x)p_j(x)\,dx.$$

Let us state the first order optimality conditions. For every perturbation h_j in the cone $\mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$, there holds $\langle dJ(a_j^*), h_j\rangle \geqslant 0$, which writes

$$-2\int_0^L h_j(x)e_{a_j^*,j}(x)p_j(x)\,dx \geqslant 0. \tag{3.20}$$

The analysis of such optimality condition is standard in optimal control theory (see for example [70]) and permits to recover easily (3.12). This ends the proof. ■

3.2.2 Proof of Theorem 3

Step 1 : existence and bang-bang property of the minimizers (first point of Theorem 3). Notice first that each of the infima defining Problem ($\mathcal{P}_{L,r,M}$) can be inverted with each other. As a result, and according to Proposition 6, there exists an optimal pair (a_j^*, ω_j^*) such that

$$\begin{aligned} m(L,M,r) &= \inf_{a\in\mathcal{A}_M(0,L)}\inf_{\omega\in\Omega_r(0,L)} J(a,\omega) = \inf_{j\in\mathbb{N}^*}\inf_{a\in\mathcal{A}_M(0,L)}\inf_{\omega\in\Omega_r(0,L)}\int_\omega e_{a,j}(x)^2\,dx \\ &= \inf_{j\in\mathbb{N}^*}\int_{\omega_j^*} e_{a_j^*,j}(x)^2\,dx. \end{aligned}$$

It remains then to prove that the last infimum is reached by some index $j_0 \in \mathbb{N}^*$. We will use the two following lemmas.

Lemma 11. *Let $M > 0$ and $(a_j)_{j\in\mathbb{N}^*}$ be a sequence of $\mathcal{A}_M(0,L)$. The sequence $(e_{a_j,j}^2)_{j\in\mathbb{N}^*}$ converges weakly-$\star$ in $L^\infty(0,L)$ to the constant function $\frac{1}{L}$.*

The proof of Lemma 11 is standard and is postponed to Appendix 3.4.2 for the sake of clarity. The proof of the next lemma can be found in [46, 82, 85].

Lemma 12. *Let $L > 0$ and $V_0 \in (0,L)$. There holds*

$$\inf_{\substack{\rho\in L^\infty(0,L;[0,1])\\ \int_0^L \rho(x)\,dx = V_0}} \int_0^L \rho(x)\sin^2\left(\frac{j\pi}{L}x\right)dx = \frac{1}{2}\left(V_0 - \frac{L}{\pi}\sin\left(\frac{\pi}{L}V_0\right)\right),$$

for every $j \in \mathbb{N}^*$. Moreover, this problem has a unique solution ρ that writes as the characteristic function of a measurable subset ω_j^* defined by $\omega_j^* = \{\sin^2(\frac{j\pi}{L}\cdot) \leqslant \eta_j\}$ for some well-chosen positive number η_j determined in such a way that the constraint $\int_0^L \rho(x)\,dx = V_0$ be satisfied.

As a consequence of Lemma 12 (which gives the last equality) and by minimality of $m(L,M,r)$ (note that $0\in\mathcal{A}_M(0,L)$), there holds

$$\begin{aligned} m(L,M,r) &= \inf_{j\in\mathbb{N}^*}\int_{\omega_j^*} e_{a_j^*,j}(x)^2\,dx = \inf_{\omega\in\Omega_r(0,L)}\inf_{j\in\mathbb{N}^*}\int_{\omega} e_{a_j^*,j}(x)^2\,dx \\ &\leqslant \inf_{\omega\in\Omega_r(0,L)}\inf_{j\in\mathbb{N}^*}\int_{\omega} e_{0,j}(x)^2\,dx = \frac{2}{L}\inf_{\omega\in\Omega_r(0,L)}\inf_{j\in\mathbb{N}^*}\int_{\omega}\sin^2\left(\frac{j\pi}{L}x\right)dx \\ &= r-\frac{1}{\pi}\sin(r\pi). \end{aligned}$$

Using Lemma 11 (the weak-$\star$ convergence being used with the "test" function $\chi_{\omega_j^*}\in L^1(0,L)$) we have $r=\lim_{j\to+\infty}\int_{\omega^*} e_{a_j^*,j}(x)^2\,dx$. Thus, $m(L,M,r)\leqslant\lim_{j\to+\infty}\int_{\omega^*} e_{a_j^*,j}(x)^2\,dx$. As a consequence, the infimum $\inf_{j\in\mathbb{N}^*}\int_{\omega_j^*} e_{a_j^*,j}(x)^2\,dx$ is reached by a finite integer j_0^*. The existence result follows.

From now on and for the sake of clarity, we will denote by (j_0,ω^*,a^*) instead of $(j_0,\omega_{j_0}^*,a_{j_0}^*)$ a solution of Problem $(\mathcal{P}_{L,r,M})$. We now prove that the solution a^* of Problem $(\mathcal{P}_{L,r,M})$ is *bang-bang*. Let us write the necessary first order optimality conditions of Problem $(\mathcal{P}_{L,r,M})$. To simplify the notations, the adjoint state introduced in Proposition 7 will be denoted by p (resp. p_i) instead of p_{j_0} (resp. $p_{j_0,i}$). For $0<\alpha<\beta<L$, introduce the sets

- $\mathcal{I}_{0,a^*}(\alpha,\beta)$: the set of elements [5] $x\in[\alpha,\beta]$ such that $a^*(x)=0$;
- $\mathcal{I}_{M,a^*}(\alpha,\beta)$: the set of elements [5] $x\in[\alpha,\beta]$ such that $a^*(x)=M$;
- $\mathcal{I}_{\star,a^*}(\alpha,\beta)$: the set of elements [5] $x\in[\alpha,\beta]$ such that $0<a^*(x)<M$, that writes also

$$\mathcal{I}_{\star,a^*}(\alpha,\beta)=\bigcup_{k=1}^{+\infty}\left\{x\in(\alpha,\beta):\frac{1}{k}<a^*(x)<M-\frac{1}{k}\right\}=:\bigcup_{k=1}^{+\infty}\mathcal{I}_{\star,a^*,k}(\alpha,\beta).$$

We will prove that the set $\mathcal{I}_{\star,a^*,k}(0,L)=\bigcup_{i_0=1}^{j}\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ has zero Lebesgue measure, for every nonzero integer k. This will imply that one has necessarily $a^*=0$ ou $a^*=M$ a.e in $(0,L)$ or in other words that is a^* is bang bang. Let us argue by contradiction, assuming that there exists $k_0\in\mathbb{N}^*$ such that $\mathcal{I}_{\star,a^*,k_0}(x_j^{i_0-1},x_j^{i_0})$ is of positive measure then $\bigcup_{k=k_0}^{+\infty}\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ is of positive measure. Let $x_0\in\bigcup_{k=k_0}^{+\infty}\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ and let $(G_{k_0,n})_{n\in\mathbb{N}}$ be a sequence of measurable subsets with $G_{k_0,n}$ included in $\bigcup_{k=k_0}^{+\infty}\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ and containing x_0. Since $a^*\in[\frac{1}{k},M-\frac{1}{k}]$ a.e in $\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ for every $k\in[k_0,\infty[$, the perturbations a^*+th and a^*-th are admissible for t small enough. Choosing $h=\chi_{G_{k_0,n}}$ and letting t go to 0, it follows that

$$\langle dJ(a^*),h\rangle=\int_{x_j^{i_0-1}}^{x_j^{i_0}} h(x)\left(-e_{a_j^*,j}(x)p_{i_0}(x)\right)dx=0\Longleftrightarrow\int_{G_{k_0,n}}\left(-e_{a_j^*,j}(x)p_{i_0}(x)\right)dx=0.$$

Dividing the last equality by $|G_{k_0,n}|$ and letting $G_{k_0,n}$ shrink to $\{x_0\}$ as $n\to+\infty$ shows that $e_{a_j^*,j}(x_0)p_{i_0}(x_0)=0$ for almost every $x_0\in\bigcup_{k=k_0}^{+\infty}\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$, according to the Lebesgue density Theorem. Since $e_{a_j^*,j}$ does not vanish on $(x_j^{i_0-1},x_j^{i_0})$ we then infer that $p_{i_0}(x)=0$ for almost every $x\in\bigcup_{k=k_0}^{+\infty}\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$. Let us prove that such a situation cannot occur. The variational formulation of System (3.13)-(3.14) writes : find $p_{i_0}\in H_0^1(x_j^{i_0-1},x_j^{i_0})$ such that for

5. up to a set of zero Lebesgue measure

every test function $\varphi \in H_0^1(x_j^{i_0-1}, x_j^{i_0})$, there holds

$$-\int_{x_j^{i_0-1}}^{x_j^{i_0}} p_{i_0}(x)\varphi''(x) + (a_{i_0}(x) - \lambda_{a_{i_0},1}^2)p_{i_0}(x)\varphi(x)\,dx = \int_{x_j^{i_0-1}}^{x_j^{i_0}} (\chi_{\omega_{j,i_0}^*} - \tilde{c}_{j,i_0})e_{a_j^*,j}\varphi(x)\,dx.$$

Since $\bigcup_{k=k_0}^{+\infty} \mathcal{I}_{\star,a^*,k}(x_j^{i_0-1}, x_j^{i_0})$ is assumed to be of positive measure, let us choose test functions φ whose support is contained in $\bigcup_{k=k_0}^{+\infty} \mathcal{I}_{\star,a^*,k}(x_j^{i_0-1}, x_j^{i_0})$. There holds

$$\int_0^L (\chi_{\omega_{j,i_0}^*} - \tilde{c}_{j,i_0})e_{a_j^*,j}(x)\varphi(x)\,dx = 0,$$

for such a choice of test functions. From Proposition 7, $\chi_{\omega_{j,i_0}^*} := \chi_{\omega_j} \cap (x_j^{i_0-1}, x_j^{i_0})$ is characterized in terms of the level set of the function $e_{a_j^*,j}^2$ and $\tilde{c}_{j,i_0} \in (0,1)$. According to the Lebesgue density Theorem and using the inner regularity of Lebesgue measurable sets, there exists a measure set K of positive measure such that $e_{a_j^*,j} = 0$ on K, whence the contradiction. We then deduce that $|\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1}, x_j^{i_0})| = 0$ and necessarily, a^* is bang-bang.

Step 2 : counting the switching points of a^* and the number of connected components of ω^* (first part of the second point of Theorem 3). For the sake of simplicity, we first give the argument in the case where the infimum $m(L,M,r)$ is reached at $j_0 = 1$ and we will then generalize our analysis to any $j_0 \in \mathbb{N}^*$.

At this step, we know that a_1^* is *bang-bang* and ω_1^* is characterized in terms of the level set of the function $e_{a_1^*,1}(\cdot)^2$. According to (3.12), the number of switching points of a_1^* corresponds to the number of zeros of the function $x \mapsto p_1(x)e_{a_1^*,1}(x)$. Let us evaluate this number.

Since $M \leqslant \pi^2/L^2$, there holds $\|a_1^*\|_\infty \leqslant \frac{\pi^2}{L^2}$. As a consequence and using (3.1), one deduces that the eigenfunction $e_{a_1^*,1}$ is concave and reaches its maximum at a unique point $x_{\max} \in (0,L)$. Moreover, since $e_{a_1^*,1}$ is increasing on $(0,x_{\max})$ and decreasing on $(x_{\max},L)$ with $e_{a_1^*,1}(0) = e_{a_1^*,1}(\pi) = 0$, from Proposition 5, there exists $(\alpha,\beta) \in (0,L)^2$ such that $\alpha < \beta$ and

$$\chi_{\omega_1^*} = 1 - \chi_{(\alpha,\beta)}, \tag{3.21}$$

with $\alpha < x_{\max} < \beta$.

Let us provide a precise description of the function p_1. One readily check by differentiating two times the function $p_1/e_{a_1^*,1}$ that the function p_1 may be written as

$$p_1(x) = f(x)e_{a_1^*,1}(x) \tag{3.22}$$

for every $x \in (0,L)$, where the function f is defined by

$$f(x) = -\int_0^x g(t)dt + f(0), \qquad \text{with} \qquad f(0) = \int_0^L \left(\int_0^x g(t)dt\right) e_{a_1^*,1}^2(x)\,dx \tag{3.23}$$

and the function g is defined by

$$g(t) = \frac{\int_0^t (\chi_{\omega_1^*}(s) - \tilde{c})e_{a_1^*,1}^2(s)\,ds}{e_{a_1^*,1}^2(t)}, \tag{3.24}$$

where here and in the rest of the proof, we will call by $\tilde{c}$ the number $\tilde{c}_{1,1}$ (whose definition was given in (3.15)). In the following result, we provide a description of the function g.

Lemma 13. *The function g defined by (3.24) verifies*

$$g(0) = g(L) = 0, \tag{3.25}$$

$$\text{there exists a unique real number } o_g \text{ in } (0, L) \text{ such that } g(o_g) = 0, \tag{3.26}$$

$$g > 0 \text{ in } (0, o_g) \text{ and } g < 0 \text{ in } (o_g, L), \tag{3.27}$$

$$g \text{ is decreasing on } (\alpha, \min(o_g, x_{max})) \text{ and } (\max(o_g, x_{max}), \beta). \tag{3.28}$$

Proof of Lemma 13. Let us first prove (3.26). We consider the function $\tilde{g}$ defined by

$$\tilde{g}(t) = \int_0^t (\chi_{\omega_1^*}(s) - \tilde{c}) e^2_{a_1^*,1}(s)\, ds, \tag{3.29}$$

so that the function g writes

$$g = \frac{\tilde{g}}{e_{a_1^*,1}(\cdot)^2}. \tag{3.30}$$

According to (3.21) and remarking that $0 < \tilde{c} < 1$ according to (3.15), the function $\tilde{g}$ is strictly increasing on $(0, \alpha)$ and (β, L), and strictly decreasing on (α, β). Besides, using (3.15), there holds

$$\tilde{g}(0) = \tilde{g}(L) = 0. \tag{3.31}$$

Hence, using the variations of $\tilde{g}$ given before and (3.31) that $\tilde{g}$ (and hence g) has a unique zero on $(0, L)$ that we call o_g from now on. Moreover, clearly $\tilde{g} > 0$ on $(0, o_g)$ and $\tilde{g} < 0$ on (o_g, L), hence, using (3.30), we deduce the same property for g and (3.27) is proved.

Equality (3.25) is readily obtained by making a Taylor expansion of $e^2_{a_1^*,1}$ and $\tilde{g}$ around 0 and L and using (3.30). Indeed, there holds

$$e^2_{a_1^*,1}(x) \sim x^2 {e^2_{a_1^*,1}}'(0) \quad \text{and} \quad \tilde{g}(x) \sim (\chi_{\omega_1^*}(0) - \tilde{c}) x^3 {e^2_{a_1^*,1}}'(0)/3 \qquad \text{as } x \to 0,$$

$$e^2_{a_1^*,1}(x) \sim \frac{(x-\pi)^2}{2} {e^2_{a_1^*,1}}'(\pi) \quad \text{and} \quad \tilde{g}(x) \sim (\chi_{\omega_1^*}(\pi) - \tilde{c})(x-\pi)^3 {e^2_{a_1^*,1}}'(\pi)/3 \qquad \text{as } x \to \pi.$$

To conclude, it remains to prove (3.28). From (3.24), one observes that g is differentiable almost everywhere on $(0, L)$ and

$$g'(x) = \chi_{\omega_1^*}(x) - \tilde{c} - \frac{2e'_{a_1^*,1}(x) g(x)}{e_{a_1^*,1}(x)}, \tag{3.32}$$

for almost every $x \in (0, L)$. Using the variations of $e_{a_1^*,1}$, (3.21) and (3.32), we infer that g' is negative almost everywhere on $(\alpha, \min(o_g, x_{\max})) \cup (\max(o_g, x_{\max}), \beta)$, from which we deduce (3.28).

The proof of Lemma 13 is then complete. ■

As a consequence of (3.27) and (3.23), f is strictly decreasing on $(0, o_g)$ and strictly increasing on (o_g, L) where o_g is defined in (3.26). We conclude that f has at most two zeros in $(0, L)$. Moreover, thanks to (3.14) and (3.22), f has at least one zero in $(0, L)$. Since $e_{a_1^*,1}(\cdot)$ does not vanish in $(0, L)$, one infers that a_1^* has at least 1 and at most 2 switching points.

To generalize our argument to any order $j \geqslant 2$, notice that using the notations of Proposition 7 and its proof, one has $(-1)^{i_0+1} e_{a^*_{j,i_0},1}(x) > 0$ for all $x \in (x_j^{i_0-1}, x_j^{i_0})$ with $i_0 \in \{1 \cdots j\}$. Then, mimicking the argument above in the particular case where $j_0 = 1$, one shows that the function

a_j^* has at most two switching points in $(x_j^{i_0-1}, x_j^{i_0})$ and at least one. Besides, since the nodal points $\{x_j^i\}_{i\in\{1,\cdots j-1\}}$ can also be switching points, we conclude that the function a_j^* has at most $3j_0-1$ and at least j_0 switching points in $(0,L)$.

Step 3 : proof of the estimate (3.6) (last part of the second point of Theorem 3). Let us first show the easier inequality, in other word the right one. It suffices to write that $m(L,M,r)$ is bounded from above by $\inf_{\omega\in\Omega_r(0,L)} J(0,\omega)$. Inverting the two infima and using Lemma 12 leads immediately to the desired estimate.

The left inequality appears more intricate to establish. It rests upon the difficult result stated in Proposition 4, whose proof is quite long and technical but the method is elementary and interesting. For this reason, we will temporarily admit this result and postpone its proof to Section 3.2.3. Let us assume for the moment that $r\leqslant \underline{r}_2$. Let us notice that $\underline{m}_2=\frac{7}{8}\sqrt{3}(3-2\sqrt{2})$ and $\underline{r}_2=\frac{\sqrt{5}}{5}+\frac{\sqrt{10}}{10}$. Then, it just remains to prove that $\underline{m}_j\geqslant \underline{m}_2$ and $\underline{r}_j\geqslant \underline{r}_2$ for every $j\in\mathbb{N}\backslash\{0,1\}$.

Proof of $\underline{m}_j\geqslant \underline{m}_2$: We introduce the function F defined on $[2,+\infty)$ by

$$F(x)=\frac{(2x^2-1)(x^2-1)^{3/2}\left(\left(\frac{x^2}{x^2-2}\right)^{1/2}-1\right)^2}{x^3\left(\left(\frac{x^2}{x^2-2}\right)^{x/2}-1\right)^2}.$$

Notice that $F(j)=3\underline{m}_j$ for every $j\in\mathbb{N}^*$. Let us write $F(x)=u(x)v(x)$ with

$$u(x)=\frac{(2x^2-1)\left(\left(\frac{x^2}{x^2-2}\right)^{1/2}-1\right)^2}{\left(\left(\frac{x^2}{x^2-2}\right)^{x/2}-1\right)^2}\quad\text{and}\quad v(x)=\frac{(x^2-1)^{3/2}}{x^3}.$$

Let us show that $u(x)\geqslant u(2)$ for every $x\geqslant 2$. This comes to show that $\psi(x)\leqslant 0$, where

$$\psi(x)=\gamma\left(\left(\frac{x^2}{x^2-2}\right)^{x/2}-1\right)-\sqrt{2x^2-1}\left(\left(\frac{x^2}{x^2-2}\right)^{1/2}-1\right),$$

with $\gamma=\sqrt{u(2)}=\sqrt{7}(\sqrt{2}-1)$, for every $x\geqslant 2$. The derivative of ψ writes

$$\psi'(x)=\gamma\left(\frac{x^2}{x^2-2}\right)^{x/2}w(x)-\frac{2x}{\sqrt{2x^2-1}}\left(\sqrt{\frac{x^2}{x^2-2}}-1\right)-\frac{\sqrt{2x^2-1}}{(x^2-2)^{3/2}},$$

with $w(x)=\ln\left(\sqrt{\frac{x^2}{x^2-2}}\right)-\frac{1}{x^2-2}$. The derivative of w writes

$$w'(x)=\frac{4}{x(x^2-2)^2}$$

and therefore, the function w is increasing and negative since $\lim_{x\to+\infty}w(x)=0$. One then infers that $\psi'(x)$ writes as the sum of three negative terms and is thus negative on $[2,+\infty)$. Hence, the function ψ decreases on this interval and therefore, $\psi(x)\leqslant\psi(2)=0$ for every $x\in[2,+\infty)$. The expected result on u follows.

Let us now show that $v(x) \geqslant v(2)$ for every $x \in [2, +\infty)$. The derivative of v writes

$$v'(x) = \frac{3\sqrt{x^2-1}}{x^4},$$

is therefore positive on $[2, +\infty)$, and the expected conclusion follows.

Proof of $\underline{r}_j \geqslant \underline{r}_2$: Let us write $\underline{r}_j = u(j)v(j)$ with

$$u(j) = \frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}} \quad \text{and} \quad v(j) = j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}.$$

We claim that $j \mapsto u(j)$ is a increasing function for every $j \geqslant 2$ and $j \mapsto v(j)$ is a increasing function for every $j \geqslant 4$. Indeed, straightforward computations show that

$$\forall j \geqslant 2, \qquad \frac{du}{dj}(j) = \frac{(j+\sqrt{j^2-2})(j^2+1-j\sqrt{j^2-2})}{2\sqrt{j^2-2}(j^2+1)^{3/2})} \geqslant 0$$

and

$$\forall j \geqslant 4, \qquad \frac{dv}{dj}(j) = \frac{(j^{j-1} + (j^2-2)^{j/2}(\ln\left(\sqrt{\gamma(j)}\right) - \gamma(j) + 1 - \frac{1}{j})}{j^{j-1}} \geqslant 0$$

We thus infer that for every $j \geqslant 4$, there holds $g(j-1, j) \geqslant g(3, 4)$. Dealing separately with the cases $j = 2$ and $j = 3$ leads to the desired conclusion for $r \leqslant \underline{r}_2$.

Let us now treat the case $r \geqslant \underline{r}_2$. It is easy to prove that Problem $\mathcal{P}_{L,r,M}$ is equivalent to considering the problem

$$\inf_{a \in \mathcal{A}_M(0,L)} \inf_{\omega \in \tilde{\Omega}_r(0,L)} J(a, \omega),$$

where

$$\tilde{\Omega}_r(\alpha, \beta) = \{\text{Lebesgue measurable subset } \omega \text{ of } (\alpha, \beta) \text{ such that } |\omega| \geqslant r(\beta - \alpha)\}.$$

Taking into account this new expression, we remark that the quantity $m(L, M, r)$ is increasing with respect to r and so for $r \geqslant \underline{r}_2$ we have $m(L, M, r) \geqslant m(L, M, \underline{r}_2) \geqslant \underline{m}_2 \underline{r}_2^3$, which concludes the proof. ■

3.2.3 Proof of Proposition 4

The proof, although quite technical, is based on a simple idea : by using the concavity of the eigenfunction $e_{a,j}$, we will determine a piecewise affine function Δ_j such that $e_{a,j} \geqslant \Delta_j$ for every $a \in \mathcal{A}_M(0, L)$. The construction of such a function is not obvious since one has to control at the same time the slope of the graph of Δ_j on each interval on which it is affine, and its maximum, as the potential a describes the set $\mathcal{A}_M(0, L)$.

As previously, we will first consider the case where $j = 1$. In other words, we will first provide a lower estimate of the quantity $m_1(L, r)$. We will then generalize this estimate to any $j \in \mathbb{N}^*$ by using that the j-th eigenfunction $e_{a,j}$ of A_a coincides with the first eigenfunction of the restriction of A_a on each nodal domain.

Notice that, proving that the estimate (3.7) holds for every $M \in (0, \pi^2/L^2]$ is equivalent to showing it for the particular value

$$M = \pi^2/L^2,$$

which will be assumed from now on. Let a be a generic element of $\mathcal{A}_M(0,L)$.

First step : proof of Proposition 4 in the case "$j=1$". By using the concavity of $e_{a,1}$ we will construct an affine function Δ_1 (see Figure 3.2) such that $e_{a,1} \geqslant \Delta_1$ pointwisely on $[0,L]$. We will infer a lower bound of $m_1(L,M,r)$ by computing explicitly the minimum of the quantity $\int_\omega \Delta_1^2(x)\,dx$ over the class of measurable subsets ω of $(0,L)$ such that $|\omega| = rL$. For that purpose, let us first provide an estimate of the $\max_{(0,L)} e_{a,1}$ in terms of the L^2 norm of $e_{a,1}$ and the derivatives of $e_{a,1}$ at $x=0$ and $x=L$.

Lemma 14. *With the assumptions of Proposition 4, there holds*

$$e_{a,1}^2(x_{max}) \geqslant \max\left\{\frac{3}{2L}\int_0^L e_{a,1}^2(x)\,dx,\, \frac{L^2}{2\pi^2}\max\{e_{a,1}'(0)^2, e_{a,1}'(L)^2\}\right\}. \tag{3.33}$$

Proof of Lemma 14. Since $e_{a,1}'(x_{max}) = 0$, multiplying Equation (3.1) by e_1' and integrating on (x, x_{max}) (with possibly $x > x_{max}$) leads to

$$e_{a,1}'(x)^2 = \int_x^{x_{max}} (\lambda_{a,j}^2 - a(x))\frac{d}{dx}(e_{a,1}^2(x))\,dx \tag{3.34}$$

for every $x \in (0,L)$. Besides, according to the Courant-Fischer minimax principle (see for instance [Section 3.4.4, (3.94)]) , one has

$$0 \leqslant \lambda_{a,1}^2 - a(\cdot) \leqslant \frac{2\pi^2}{L^2} \qquad \text{in } (0,L). \tag{3.35}$$

Therefore, combining (3.34) and (3.35) yields

$$e_{a,1}'(x)^2 \leqslant \frac{2\pi^2}{L^2}(e_{a,1}^2(x_{max}) - e_{a,1}^2(x)), \tag{3.36}$$

for every $x \in (0,L)$. In particular, applying (3.36) at $x=L$ and $x=0$, we obtain

$$\max\{e_{a,1}'(L)^2, e_{a,1}'(0)^2\} \leqslant \frac{2\pi^2}{L^2}e_{a,1}^2(x_{max}), \tag{3.37}$$

Moreover, by integrating (3.36) between 0 and L, we obtain

$$\int_0^L e_{a,1}'(x)^2\,dx + \frac{2\pi^2}{L^2}\int_0^L e_{a,1}^2(x)\,dx \leqslant \frac{2\pi^2}{L^2}e_{a,1}^2(x_{max})L. \tag{3.38}$$

We obtain (3.33) from (3.37) and by combining (3.38) with Poincaré's inequality

$$\int_0^L e_{a,1}(x)^2\,dx \leqslant \frac{L^2}{\pi^2}\int_0^L e_{a,1}'(x)^2\,dx.$$

■

According to (3.33), and assuming since the eigenfunction $e_{a,1}$ is normalized in $L^2(0,L)$, there holds

$$e_{a,1}^2(x_{max}) \geqslant \frac{3}{2L}. \tag{3.39}$$

Since $e_{a,1}$ is concave and according to (3.39), one has the successive inequalities

$$e_{a,1}(x) \geqslant Tr_{a,1}(x) \geqslant \triangle_1(x), \tag{3.40}$$

for every $x \in [0, L]$, where $Tr_{a,1}$ and $\triangle_1$ denote the piecewise affine functions defined by

$$Tr_{a,1}(x) = \begin{cases} \frac{e_{a,j}(x_{max})x}{x_{max}} & \text{on } (0, x_{max}) \\ \frac{e_{a,j}(x_{max})(L-x)}{L-x_{max}} & \text{on } (x_{max}, L) \end{cases} \quad \text{and} \quad \triangle_1(x) = \begin{cases} \frac{\sqrt{3}x}{\sqrt{2L}x_{max}} & \text{on } (0, x_{max}) \\ \frac{\sqrt{3}(L-x)}{\sqrt{2L}(L-x_{max})} & \text{on } (x_{max}, L). \end{cases}$$

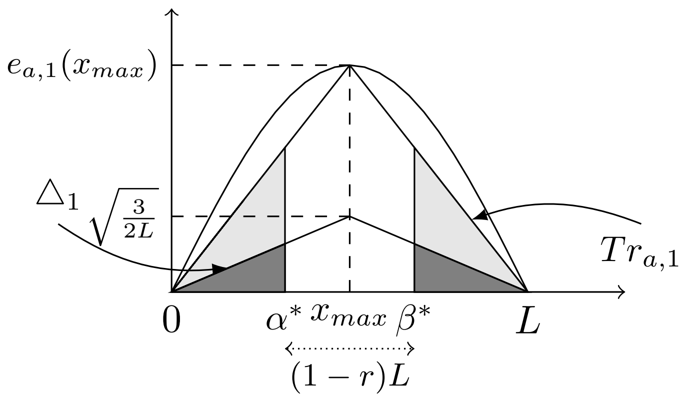

FIGURE 3.2 – Graphs of the functions $e_{a,1}$, $Tr_{a,1}$ and $\triangle_1$.

Combining (3.39) with (3.40) and according to Proposition 5, we readily obtain

$$\inf_{\omega \in \Omega_r(0,L)} \int_\omega e_{a,1}(x)^2\, dx \geqslant \inf_{\omega \in \Omega_r(0,L)} \int_\omega \triangle_1(x)^2 dx = \int_{\widehat{\omega}} \triangle_1(x)^2 dx, \tag{3.41}$$

with $\widehat{\omega} = (0, \alpha^*) \cup (\beta^*, L)$ verifying

$$\triangle_1(\alpha^*) = \triangle_1(\beta^*) \qquad \text{and} \qquad |\widehat{\omega}| = L - \beta^* + \alpha^* = rL.$$

It follows that $\alpha^* = rx_{max}$, $\beta^* = (1-r)L + rx_{max}$ and one obtains

$$\int_{\widehat{\omega}} \triangle_1(x)^2\, dx = \int_0^{rx_{max}} \triangle_1(x)^2\, dx + \int_{(1-r)L+rx_{max}}^L \triangle_1(x)^2\, dx = \frac{r^3}{2} = r^3 \underline{m}_1. \tag{3.42}$$

We have then proved Proposition 4 in the case where $j = 1$.

Second step : proof of Proposition 4 in the general case. In what follows, we assume that $j \geqslant 2$.

Let $0 = x_j^0 < x_j^1 < x_j^2 < \cdots < x_j^{j-1} < L = x_j^j$ be the $j+1$ zeros of the j-th eigenfunction $e_{a,j}$.

Introduce $\gamma = (\gamma_1, \cdots, \gamma_{j-1}) \in (0,1)^j$ such that

$$\gamma_i = \int_{x_j^i}^{x_j^{i+1}} e_{a,j}^2(x)\,dx, \qquad i = 0, \dots, j-1. \tag{3.43}$$

Note that, because of the normalization condition on the function $e_{a,j}$, there holds

$$\sum_{i=0}^{j-1} \gamma_i = 1. \tag{3.44}$$

In the sequel, we will use the following notations, for all $i \in \{0, \cdots, j-1\}$,

$$\Omega_i = (x_j^i, x_j^{i+1}), \qquad \eta_i = \frac{|\Omega_i|}{L} \in (0,1), \qquad \text{and} \qquad e_{a,j}(x_{max}^i) = \max_{x\in\Omega i} e_{a,j}(x). \tag{3.45}$$

Now, assume that $r \leqslant \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right)$. We will distinguish between several cases, depending on the value of the first integer $i_0 \in \{0, \cdots, j-1\}$ (that exists thanks to (3.44)) such that

$$\gamma_{i_0} \geqslant \frac{1}{j}. \tag{3.46}$$

By exploiting condition (3.46), we will yield a lower bound A_{i_0} of the positive number $|e_{a,j}(x_{max}^{i_0})|$. Then we will derive an estimate of $e_{a,j}(x_{max}^{i_0-1})$ and $e_{a,j}(x_{max}^{i_0+1})$ in terms of $e_{a,j}(x_{max}^{i_0})$. Hence, step by step we will get lower bounds of all j maxima needed to construct the piecewise affine function Δ_j. For that purpose, we have to distinguish between several cases, depending on the value of the integer i_0

<u>*First case :*</u> assume that $i_0 = 0$. Since the function $e_{a,j}(\cdot)$ is concave, we claim that

$$\int_\omega e_{a,j}(x)^2\,dx \geqslant \int_\omega Tr_{a,j}(x)^2 dx, \tag{3.47}$$

for every $\omega \in \Omega_r(0,L)$ where the function $Tr_{a,j}$ is piecewise affine, defined on each interval (x_j^i, x_j^{i+1}), with $i \in \{0, \dots, j-1\}$, by

$$Tr_{a,j}(x) = \begin{cases} \frac{x-x_j^i}{x_{max}^i - x_j^i} e_{a,j}(x_{max}^i) & \text{on } (x_j^i, x_{max}^i), \\ \frac{x_j^{i+1}-x}{x_j^{i+1}-x_{max}^i} e_{a,j}(x_{max}^i) & \text{on } (x_{max}^i, x_j^{i+1}), \end{cases}$$

for every $x \in (x_j^i, x_j^{i+1})$.

Since the j-th eigenfunction $e_{a,j}$ coincides with the first eigenfunction of $-\partial_{xx} + a(\cdot)$ with Dirichlet conditions on (x_j^i, x_j^{i+1}), the method of the first step can be adapted. By reproducing the proof of Lemma 14 to show (3.33), we obtain

$$e_{a,j}(x_{max}^i)^2 \geqslant \max\left\{ \frac{\frac{2\pi^2}{|\Omega_i|^2} + \frac{\pi^2}{L^2}}{\left(\frac{\pi^2}{|\Omega_i|^2} + \frac{\pi^2}{L^2}\right)|\Omega_i|} \int_{x_j^i}^{x_j^{i+1}} e_{a,j}(x)^2\,dx, \frac{1}{\frac{\pi^2}{|\Omega_i|^2} + \frac{\pi^2}{L^2}} \max\{e'_{a,j}(x_j^i)^2, e'_{a,j}(x_j^{i+1})^2\} \right\}. \tag{3.48}$$

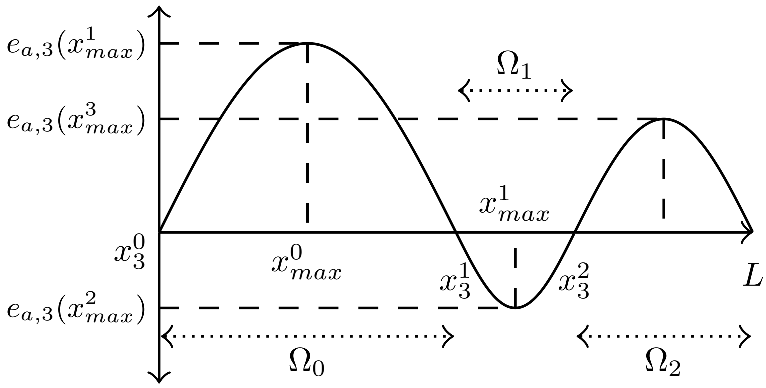

FIGURE 3.3 – Illustration of the case “$i_0 = 0$” with 3 nodal domains ($j = 3$).

Notice that one recovers (3.33) by substituting $|\Omega_i|$ by L in (3.48). One derives from the equivalent of (3.36) in this case the estimates

$$\frac{\pi^2}{|\Omega_i|^2} - \frac{\pi^2}{L^2} \leqslant \frac{e'_{a,j}(x^i_j)^2}{e_{a,j}(x^i_{max})^2} \leqslant \frac{\pi^2}{|\Omega_i|^2} + \frac{\pi^2}{L^2} \quad \text{and} \quad \frac{\pi^2}{|\Omega_i|^2} - \frac{\pi^2}{L^2} \leqslant \frac{e'_{a,j}(x^{i+1}_j)^2}{e_{a,j}(x^i_{max})^2} \leqslant \frac{\pi^2}{|\Omega_i|^2} + \frac{\pi^2}{L^2}. \tag{3.49}$$

Applying (3.48) for $i = 0$ and using (3.46) yields

$$e_{a,j}(x^0_{max})^2 \geqslant A_0 \quad \text{with} \quad A_0 = \frac{\frac{2\pi^2}{|\Omega_0|^2} + \frac{\pi^2}{L^2}}{j\left(\frac{\pi^2}{|\Omega_0|^2} + \frac{\pi^2}{L^2}\right)|\Omega_0|} = \frac{1}{Lj}\left(\frac{2+\eta_0^2}{\eta_0(1+\eta_0^2)}\right). \tag{3.50}$$

Let us now provide an estimate of $e_{a,1}(x^1_{max})^2$. Combining the inequalities (3.48) with $i = 1$ and (3.49) with $i = 0$, we get

$$e_{a,1}(x^1_{max})^2 \geqslant \frac{e'_{a,j}(x^1_j)^2}{\frac{\pi^2}{|\Omega_1|^2} + \frac{\pi^2}{L^2}} \geqslant \frac{(\frac{\pi^2}{|\Omega_0|^2} - \frac{\pi^2}{L^2})e_{a,1}(x^0_{max})^2}{\frac{\pi^2}{|\Omega_1|^2} + \frac{\pi^2}{L^2}}. \tag{3.51}$$

Combining (3.50) and (3.51) yields

$$e_{a,j}(x^1_{max})^2 \geqslant A_1 \quad \text{with} \quad A_1 = \frac{(L^2 - |\Omega_0|^2)|\Omega_1|^2}{(L^2 + |\Omega_1|^2)|\Omega_0|^2} A_0 = \frac{(1-\eta_0^2)\eta_1^2}{(1+\eta_1^2)\eta_0^2} A_0.$$

By induction, it follows that

$$e_{a,j}(x^i_{max})^2 \geqslant A_i \quad \text{with} \quad A_i = \left(\prod_{k=1}^{i} \frac{(1-\eta_{k-1}^2)\eta_k^2}{(1+\eta_k^2)\eta_{k-1}^2}\right) A_0, \tag{3.52}$$

for every $i \in \{1, \cdots, j-1\}$. Hence, (3.52) together with (3.47) allows us to write

$$\int_\omega e_{a,j}(x)^2\,dx \geqslant \int_\omega \triangle_j(x)^2\,dx,$$

where $\triangle_j$ is the piecewise affine function defined on $(0, L)$ by

$$\triangle_j(x) = \begin{cases} \frac{(x-x_j^i)}{(x_{max}^i - x_j^i)}\sqrt{A_i} & \text{on } (x_j^i, x_{max}^i), \\ \frac{(x_j^{i+1}-x)}{(x_j^{i+1}-x_{max}^i)}\sqrt{A_i} & \text{on } (x_{max}^i, x_j^{i+1}), \end{cases} \tag{3.53}$$

for every $i \in \{0, \cdots, j-1\}$ and $x \in (x_j^i, x_j^{i+1})$.

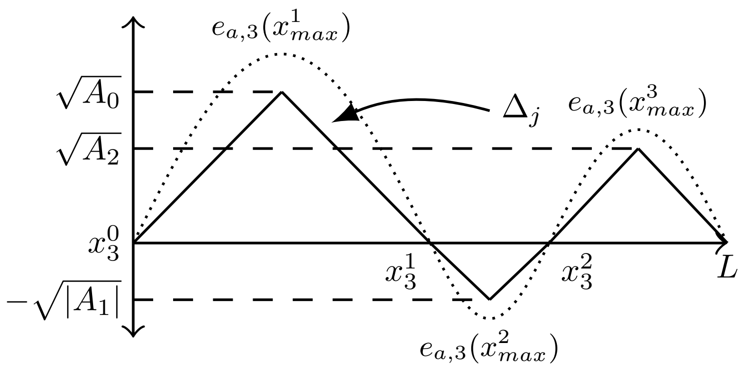

FIGURE 3.4 – Graphs of the functions $e_{a,1}$ and $\triangle_j$.

According to Proposition 5, we obtain

$$\inf_{\omega \in \Omega_r(0,L)} \int_\omega e_{a,j}(x)^2\, dx \geqslant \inf_{\omega \in \Omega_r(0,L)} \int_\omega \triangle_j(x)^2\, dx = \int_{\hat{\omega}} \triangle_j(x)^2\, dx, \tag{3.54}$$

where $\hat{\omega} = \{\triangle_j(x)^2 < \tau\}$ up to a set of zero Lebesgue measure and $|\hat{\omega}| = rL$. Let $i \in \{0, \cdots, j-1\}$ and let us introduce $\omega_i = \hat{\omega} \cap (x_j^i, x_j^{i+1})$.

The following technical Lemma allows to deal with small values of the parameter r.

Lemma 15. *Let $j \geqslant 1$, $L, \tau > 0$, let $r \in (0,1)$ such that $|\hat{\omega}| = rL$, where $\hat{\omega} = \{\triangle_j^2(\cdot) < \tau\}$, with $\triangle_j$ defined by (3.53). Recall that A_0 is defined by (3.50) and A_i is defined by (3.52) for every $i \in \{1, \cdots, j-1\}$. There holds*

$$r < \left(\frac{j + \sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right) \quad \Longrightarrow \quad \tau \leqslant \min_{i \in \{0,\cdots,j-1\}} A_i.$$

The proof Lemma 15 is postponed to Section 3.4.3. As a consequence of this lemma, there exist α_i^*, β_i^* and $r_i \in (0,1)$ such that $\omega_i = (x_j^i, \alpha_i^*) \cup (\beta_i^*, x_j^{i+1})$, $|\omega_i| = r_i(x_j^{i+1} - x_j^i)$ (see Figure 3.5 for an illustration), and therefore

$$\sum_{i=0}^{j-1} r_i(x_j^{i+1} - x_j^i) = rL. \tag{3.55}$$

By definition of $\hat{\omega}$, one has $\triangle_j^2(\alpha_i^*) = \triangle_j^2(\beta_i^*) = \triangle_j^2(\alpha_{i+1}^*)$, consequently there holds

$$\alpha_i^* = r_i(x_{max}^i - x_j^i) + x_j^i \quad , \quad \beta_i^* = x_j^{i+1} - r_i(x_j^{i+1} - x_{max}^i), \tag{3.56}$$

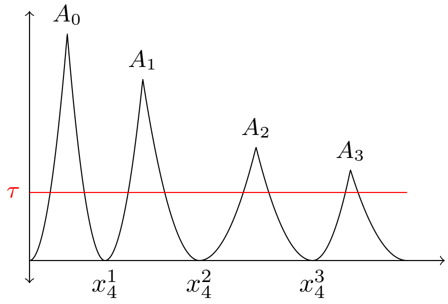

FIGURE 3.5 – $j = 4$. Graph of Δ_4^2 with respect to x.

and

$$r_{i+1}^2 A_{i+1} = r_i^2 A_i = \cdots = r_0^2 A_0. \tag{3.57}$$

As a result, one obtains

$$\int_{\omega^*} \triangle_j(x)^2\, dx = \sum_{i=0}^{j-1} \int_{\omega_i} \triangle_j(x)^2\, dx = \frac{1}{3} \sum_{i=0}^{j-1} r_i^3 |\Omega_i| A_i. \tag{3.58}$$

To compute the numbers r_i, we use (3.57) together with (3.52), which yields to

$$r_i = \sqrt{\frac{A_0}{A_i}} r_0 = \sqrt{\prod_{k=1}^{i} \frac{(1+\eta_k^2)\eta_{k-1}^2}{(1-\eta_{k-1}^2)\eta_k^2}}\, r_0. \tag{3.59}$$

Since $\sum_{i=0}^{j-1} r_i \frac{|\Omega_i|}{L} = \sum_{i=0}^{j-1} r_i \eta_i = r$, one infers

$$\begin{aligned} r_0 &= \frac{r}{\eta_0 + \sum_{i=1}^{j-1} \eta_i \sqrt{\prod_{k=1}^{i} \frac{(1+\eta_k^2)\eta_{k-1}^2}{(1-\eta_{k-1}^2)\eta_k^2}}} \\ &= \frac{r}{\eta_0 \left(1 + \sum_{i=1}^{j-1} \sqrt{\prod_{k=1}^{i} \frac{(1+\eta_k^2)}{(1-\eta_{k-1}^2)}}\right)}. \end{aligned} \tag{3.60}$$

Besides, using (3.59), there holds

$$\sum_{i=0}^{j-1} r_i^3 |\Omega_i| A_i = r_0^2 A_0 \sum_{i=0}^{j-1} r_i |\Omega_i| = r_0^2 A_0 r L. \tag{3.61}$$

We conclude by combining (3.61) with (3.58) that

$$\int_{\hat{\omega}} \triangle_j(x)^2\, dx = \frac{1}{3} A_0 r_0^2 r L, \tag{3.62}$$

where A_0 and r_0 are respectively given by (3.50) and (3.60).

Since our goal is to estimate $\int_{\hat{\omega}} \triangle_j(x)^2\, dx$ from below, regarding (3.62), we need to find a lower bound on r_0 and consequently on the numbers $|\Omega_i|$ according to (3.61). We will use the following Lemma.

Lemma 16. *For every $i \in \{0, \cdots, j-1\}$, there holds*

$$\frac{L}{\sqrt{j^2+1}} \leqslant |\Omega_i| \leqslant \frac{L}{\sqrt{j^2-1}}. \tag{3.63}$$

Proof of Lemma 16. According to the Courant-Fischer minimax principle, one has

$$\frac{j\pi}{L} \leqslant \lambda_{a,j} \leqslant \sqrt{\left(\frac{j\pi}{L}\right)^2 + \frac{\pi^2}{L^2}}.$$

Since the j-th eigenfunction $e_{a,j}$ is also the first eigenfunction of the operator $-\partial_{xx} + a(\cdot)$ with Dirichlet boundary conditions on Ω_i, we also have

$$\frac{\pi}{|\Omega_i|} \leqslant \lambda_{a,j} \leqslant \sqrt{\left(\frac{\pi}{|\Omega_i|}\right)^2 + \frac{\pi^2}{L^2}}.$$

We then infer

$$\frac{\pi}{\sqrt{\left(\frac{j\pi}{L}\right)^2 + \frac{\pi^2}{L^2}}} \leqslant |\Omega_i| \leqslant \frac{\pi}{\sqrt{\left(\frac{j\pi}{L}\right)^2 - \frac{\pi^2}{L^2}}}.$$

■

It follows from Lemma 16 that $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$ and therefore

$$\sum_{i=1}^{j-1} \sqrt{\prod_{k=1}^{i} \frac{(1+\eta_k^2)}{(1-\eta_{k-1}^2)}} \leqslant g_1(j),$$

where

$$g_1(j) = \sum_{i=1}^{j-1} \left(\frac{j^2}{j^2-2}\right)^{\frac{i}{2}} = \frac{\left(\frac{j^2}{j^2-2}\right)^{\frac{j}{2}} - \left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}}{\left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}} - 1}.$$

According to (3.60), one has

$$r_0 \geqslant \frac{r}{\eta_0(1+g_1(j))}. \tag{3.64}$$

Combining (3.50), (3.62) and (3.64), we obtain

$$\int_{\hat{\omega}} \triangle_j(x)^2\, dx \geqslant r^3 \inf_{\eta_0 \in \left(\frac{1}{\sqrt{j^2+1}}, \frac{1}{\sqrt{j^2-1}}\right)} g_2(\eta_0, j), \tag{3.65}$$

with

$$g_2(\eta_0, j) = \frac{1}{3j}\left(\frac{2+\eta_0^2}{\eta_0(1+\eta_0^2)}\right)\left(\frac{1}{\eta_0+\eta_0 g_1(j)}\right)^2.$$

Since for every $\eta_0 > 0$, we have

$$\frac{\partial g_2}{\partial \eta_0}(\eta_0, j) = -\frac{\left(\sqrt{\frac{j^2}{j^2-1}} - 1\right)^2 (11\eta_0^2 + 3\eta_0^4 + 6)}{\left((\frac{j^2}{j^2-1})^{\frac{j}{2}} - 1\right)^2 \eta_0^4 (1+\eta_0^2)^2 j} \leqslant 0,$$

the function $\eta_0 \mapsto g_2(\eta_0, j)$ is decreasing, so that (3.65) becomes

$$\int_{\hat{\omega}} \triangle_j(x)^2\, dx \geqslant r^3 g_2\left(\sqrt{\frac{1}{j^2-1}}, j\right) = r^3 \underline{m}_j, \tag{3.66}$$

and the expected result is proved for $r \leqslant \underline{r}_j := \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right)$.

Noticing that $r \mapsto \inf_{\omega\in\Omega_r(0,L)} \int_\omega \triangle_j(x)^2 dx$ is an increasing function, we infer that for every $r \in [\underline{r}_j, 1]$, there holds

$$\inf_{\omega\in\Omega_r(0,L)} \int_\omega \triangle_j(x)^2\, dx \geqslant \inf_{\omega\in\Omega_{\underline{r}_j}(0,L)} \int_\omega \triangle_j(x)^2 dx \geqslant \underline{r}_j^3 \underline{m}_j,$$

and the expected result is proved for $r \in [0, 1]$.

<u>*Second case :*</u> assume now that $i_0 = 1$. We will prove that the estimate choosing $i_0 = 0$ is worst than the estimate that we obtain with $i_0 = 1$. Using (3.48) with $i = 1$, we have

$$e_{a,j}^2(x_{max}^1) \geqslant A_1 \quad \text{with} \quad A_1 = \frac{1}{Lj}\left(\frac{2+\eta_1^2}{\eta_1(1+\eta_1^2)}\right). \tag{3.67}$$

Combining the inequality (3.48) with $i = 0$, (3.49) with $i = 1$ and (3.67) we get

$$A_0 = \frac{(1-\eta_1^2)\eta_0^2}{(1+\eta_0^2)\eta_1^2} A_1. \tag{3.68}$$

Using (3.48) with $i = 2$, (3.49) with $i = 1$ and (3.67) we have

$$e_{a,j}^2(x_{max}^2) \geqslant A_2 \quad \text{with} \quad A_2 = \frac{(1-\eta_1^2)\eta_2^2}{(1+\eta_2^2)\eta_1^2} A_1.$$

By induction, for every $i \in \{2, \cdots, j-1\}$ we have

$$e_{a,j}^2(x_{max}^i) \geqslant A_i \quad \text{with} \quad A_i = \left(\prod_{k=2}^{i} \frac{(1-\eta_{k-1}^2)\eta_k^2}{(1+\eta_k^2)\eta_{k-1}^2}\right) A_1 \tag{3.69}$$

Let us state the equivalent of Lemma 15 for the case considered here.

Lemma 17. *Let $j \geqslant 1$, $L, \tau > 0$, let $r \in (0,1)$ such that $|\hat{\omega}| = rL$, where $\hat{\omega} = \{\triangle_j^2(\cdot) < \tau\}$, with $\triangle_j$ is defined by (3.53). Recall that A_1 is defined by (3.67), A_0 is defined by (3.68) and A_i is defined by (3.69) for every $i \in \{2, \cdots, j-1\}$. There holds*

$$r < \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right) \quad \Longrightarrow \quad \tau \leqslant \min_{i\in\{0,\cdots,j-1\}} A_i.$$

The proof of this lemma is postponed to Section 3.4.3.

Hence, we conclude that $r_i^2 A_i = r_1^2 A_1$ for all $i \in \{0, \cdots, j-1\}$. Since $\sum_{i=0}^{j-1} r_i \eta_i = r$, one computes by using (3.69) and (3.68)

$$\begin{aligned} r_1 &= \frac{r}{\eta_1 + \eta_0 \sqrt{\frac{A_1}{A_0}} + \sum_{i=2}^{j-1} \eta_i \sqrt{\frac{A_1}{A_i}}} \\ &= \frac{r}{\eta_1 + \eta_0 \sqrt{\frac{1+\eta_0^2}{1-\eta_1^2}} + \sum_{i=2}^{j-1} \eta_i \sqrt{\prod_{k=2}^{i} \frac{(1+\eta_k^2)\eta_{k-1}^2}{(1-\eta_{k-1}^2)\eta_k^2}}}. \end{aligned}$$

Moreover,

$$\sum_{i=2}^{j-1} \eta_i \sqrt{\prod_{k=2}^{i} \frac{(1+\eta_k^2)\eta_{k-1}^2}{(1-\eta_{k-1}^2)\eta_k^2}} = \eta_1 \sum_{i=2}^{j-1} \sqrt{\prod_{k=2}^{i} \frac{1+\eta_k^2}{1-\eta_{k-1}^2}}$$

and since $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$ according to Lemma 16, there holds

$$r_1 \geqslant \frac{r}{\eta_1 + \eta_1 \sqrt{\frac{j^2}{j^2-2}} + \eta_1 \sum_{i=2}^{j-1} \left(\frac{j^2}{j^2-2}\right)^{\frac{i-1}{2}}}.$$

Since $j \geqslant 2$, we have

$$\sqrt{\frac{j^2}{j^2-2}} + \sum_{i=2}^{j-1} \left(\frac{j^2}{j^2-2}\right)^{\frac{i-1}{2}} - \sum_{i=1}^{j-1} \left(\frac{j^2}{j^2-2}\right)^{\frac{i}{2}} = \sqrt{\frac{j^2}{j^2-2}} - \left(\frac{j^2}{j^2-2}\right)^{\frac{j-1}{2}} \leqslant 0,$$

and it follows that

$$r_1 \geqslant \frac{r}{\eta_1 + \eta_1 \sum_{i=1}^{j-1} \left(\frac{j^2}{j^2-2}\right)^{\frac{i}{2}}}.$$

As a consequence, using the same approach as the one used for the case where $i_0 = 0$, we infer that

$$\inf_{\omega \in \Omega_r(0,L)} \int_\omega \triangle_j(x)^2 \, dx \geqslant r^3 \inf_{\eta_1 \in \left(\frac{1}{\sqrt{j^2+1}}, \frac{1}{\sqrt{j^2-1}}\right)} g_2(\eta_1, j)$$

with

$$g_2(\eta_1, j) = \frac{1}{3j} \left(\frac{2+\eta_1^2}{\eta_1(1+\eta_1^2)}\right) \left(\frac{1}{\eta_1 + \eta_1 g_1(j)}\right)^2 \quad \text{and} \quad g_1(j) = \frac{\left(\frac{j^2}{j^2-2}\right)^{\frac{j}{2}} - \left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}}{\left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}} - 1}.$$

Noticing that the functions g_1 and g_2 are exactly the same as in the case $i_0 = 1$, we conclude similarly to the first case.

Finally, mimicking this proof and adapting it for every $i_0 \in \{2, \cdots, j-1\}$, we prove that the estimate with $i_0 = 0$ is the worst one. We then obtain the same conclusion.

■

3.2.4 Proof of Theorem 4

We argue by contradiction, assuming that the optimal design problem ($\mathcal{P}_{j,L,r,\infty}$) has a solution $a^* \in L^\infty(0,L)$. Then, there exists M_0 such that a^* is a solution of the problem

$$\inf_{a\in\mathcal{A}_\infty(0,L)} \inf_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j}(x)^2\,dx = \inf_{a\in\mathcal{A}_{M_0}(0,L)} \inf_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j}(x)^2\,dx.$$

We will use the notations of Proposition 7 and Section 3.2.2.

The contradiction will be obtained by constructing a perturbation a_n^* of a^* such that

$$J(a_n^*) < J(a^*).$$

According to Proposition 7, a^* is non-trivial and *bang-bang*, equal to 0 and M_0 almost everywhere in $(0,L)$ so that there exists $i_0 \in \{1,\cdots,j\}$ such that the set $\mathcal{I}_{M_0,a^*}(x_j^{i_0-1}, x_j^{i_0})$ is measurable of positive measure.

Thanks to the regularity of the Lebesgue measure, there exists an increasing sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ strictly included in $\mathcal{I}_{M_0,a^*}(x_j^{i_0-1}, x_j^{i_0})$ satisfying

$$\lim_{n\to\infty} |K_n| = |\mathcal{I}_{M_0,a^*}(x_j^{i_0-1}, x_j^{i_0})|,$$

where $|\cdot|$ denotes the Lebesgue measure. In what follows, we will use the notation I^c to denote the complement of any set $I \subset [0,L]$ in $[0,L]$. We introduce

$$a_n^*(x) = \begin{cases} M_0 + \frac{1}{\varphi(n)} & \text{on } K_n, \\ 0 & \text{on } K_n^c, \end{cases} \quad \text{with} \quad \varphi(n) = |\mathcal{I}_{M_0,a^*}(x_j^{i_0-1}, x_j^{i_0}) \cap K_n^c|.$$

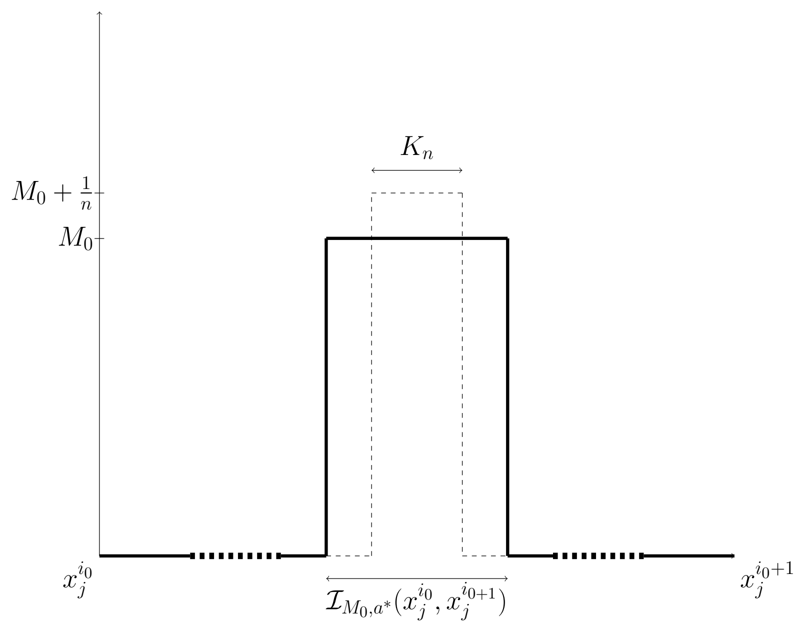

FIGURE 3.6 – perturbation $a_n^*(\dots)$ of $a^*(-)$.

Let us remark that

$$a_n^*(x) - a(x) = \begin{cases} \frac{1}{\varphi(n)} & \text{on } K_n, \\ -M_0 & \text{on } \mathcal{I}_{M_0,a^*}(x_j^{i_0-1}, x_j^{i_0}) \cap K_n^c, \\ 0 & \text{on } \mathcal{I}_{M_0,a^*}(x_j^{i_0-1}, x_j^{i_0})^c. \end{cases}$$

Hence, we get

$$\begin{aligned} \langle dJ(a), \varphi(n)(a_n^* - a^*)\rangle &= -2\int_{x_j^{i_0-1}}^{x_j^{i_0}} \varphi(n)(a_n^*(x) - a^*(x)) e_{a_{i_0},1}(x) p_{i_0}(x)\, dx \\ &= 2M_0\varphi(n) \int_{\mathcal{I}_{M_0,a^*}(x_j^{i_0-1}, x_j^{i_0}) \cap K_n^c} e_{a_{i_0},1}(x) p_{i_0}(x)\, dx \\ &\quad - \int_{K_n} e_{a_{i_0},1}(x) p_{i_0}(x)\, dx, \end{aligned}$$

for $n \in \mathbb{N}$. Using (3.12), we have $e_{a_{i_0},1} p_{i_0} \geqslant 0$ on $\mathcal{I}_{M_0,a^*}(x_j^{i_0-1}, x_j^{i_0}) \cap K_n^c$ and $e_{a_{i_0},1} p_{i_0} > 0$ on K_n for all $n \in \mathbb{N}$. Since $\lim_{n\to+\infty} \varphi(n) = 0$ and according to the Lebesgue density theorem,

$$\lim_{n\to+\infty} 2M_0\varphi(n) \int_{\mathcal{I}_{M_0,a^*}(x_j^{i_0-1}, x_j^{i_0}) \cap K_n^c} e_{a_{i_0},1}(x) p_{i_0}(x)\, dx = 0.$$

As a consequence, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$

$$\langle dJ(a), \varphi(n)(a_n^* - a^*)\rangle < 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ verifying $J(a_{n_1}^*) < J(a^*)$, whence the contradiction. We then infer that the optimal design problem ($\mathcal{P}_{j,L,r,\infty}$) has no solution.

3.3 Applications and numerical investigations

3.3.1 Controllability issues for the wave equation

3.3.1.1 The cost of the control in large time

Let us fix $T > 0$ and consider the one dimensional wave equation with potential

$$\begin{aligned} \partial_{tt}\varphi(t,x) - \partial_{xx}\varphi(t,x) + a(x)\varphi(t,x) = 0, & \qquad (t,x) \in (0,T) \times (0,L), \\ \varphi(t,0) = \varphi(t,L) = 0, & \qquad t \in [0,T], \\ (\varphi(0,x), \partial_t\varphi(0,x)) = (\varphi_0(x), \varphi_1(x)), & \qquad x \in [0,L], \end{aligned} \tag{3.70}$$

where the potential $a(\cdot)$ is a nonnegative function belonging to $L^\infty(0,L)$. It is well known that for every initial data $(\varphi_0, \varphi_1) \in H_0^1(0,L) \times L^2(0,L)$, there exists a unique solution φ in $C^0(0,T; H_0^1(0,L)) \cap C^1(0,T; L^2(0,L))$ of the Cauchy problem (3.70).

Let ω be a given measurable subset of $(0,L)$ of positive Lebesgue measure. The system (3.70) is said to be *observable* on ω in time T if there exists a positive constant c such that

$$cE_a(0) \leqslant \int_0^T \int_\omega \partial_t\varphi(t,x)^2\, dxdt \tag{3.71}$$

for all $(\varphi_0, \varphi_1) \in H_0^1(0,L) \times L^2(0,L)$ where

$$E_a(t) = \int_0^L \left(\partial_t \varphi(t,x)^2 + \partial_x \varphi(t,x)^2 + a(x)\varphi(t,x)^2\right) dx$$

for all $t \geqslant 0$. Notice moreover that the function $E_a(\cdot)$ is constant [6]. We denote by $c(T,a,\omega)$ the largest constant in the previous inequality, that is

$$c(T,a,\omega) = \inf_{\substack{(\varphi_0,\varphi_1)\in H_0^1(0,L)\times L^2(0,L)\\ (\varphi_0,\varphi_1)\neq(0,0)}} \frac{\int_0^T \int_\omega \partial_t \varphi(t,x)^2 \, dxdt}{E_a(0)}. \tag{3.72}$$

This constant can be interpreted a quantitative measure of the well-posed character of the inverse problem of reconstructing the solutions from measurements over $[0,T]\times\omega$. Moreover, this constant also plays a crucial role in the frameworks of control theory. Indeed, consider the internally controlled wave equation on $(0,L)$ with Dirichlet boundary conditions

$$\begin{cases} \partial_{tt} y(t,x) - \partial_{xx} y(t,x) + a(x) y(t,x) = u(t,x), & (t,x) \in (0,T)\times(0,L), \\ y(t,0) = y(t,\pi) = 0, & t \in [0,T], \\ (y(0,x), \partial_t y(0,x)) = (y^0(x), y^1(x)), & x \in (0,L), \end{cases} \tag{3.73}$$

where u is a control supported by $[0,T]\times\omega$ and ω is a Lebesgue measurable subset of $(0,L)$. Recall that for every initial data $(y^0,y^1) \in L^2(0,L)\times H^{-1}(0,L)$ and every $u \in L^2((0,T)\times(0,L))$, the problem (3.73) has a unique solution y verifying moreover $y \in C^0(0,T;L^2(0,L)) \cap C^1(0,T;H^{-1}(0,L))$. This problem is said to be *null controllable at time* T if and only if for every initial data $(y^0,y^1) \in L^2(0,L)\times H^{-1}(0,L)$, one can find a control $u \in L^2((0,T)\times(0,L))$ supported by $[0,T]\times\omega$ such that the solution y of (3.73) verifies $y(T,\cdot) = \partial_t y(T,\cdot) = 0$.

Let us assume that (3.73) is null controllable. At fixed $(y^0,y^1) \in L^2(0,L)\times H^{-1}(0,L)$, since the set of all controls u steering (y^0,y^1) to $(0,0)$ is a closed vector space of $L^2((0,T)\times(0,L))$, there exists a unique control of minimal $L^2((0,T)\times\omega)-$norm that we denote $u_{\min}$, which can be constructed "explicitly" as the minimum of a functional according to the Hilbert Uniqueness Method (HUM, see [73]). Thus, we can define the HUM operator Γ_ω by

$$\begin{array}{rrcl} \Gamma_\omega: & H_0^1(0,L)\times L^2(0,L) & \longrightarrow & L^2((0,T)\times(0,L)) \\ & (y^0,y^1) & \longmapsto & u_{\min}. \end{array}$$

Γ_ω is linear and continuous and we define its norm

$$\|\Gamma_\omega\| = \sup\left\{ \frac{\|u_{min}\|_{L^2((0,T)\times(0,L))}}{\|(y^0,y^1)\|_{L^2(0,L)\times H^{-1}(0,L)}} \mid (y^0,y^1) \in L^2(0,L)\times H^{-1}(0,L)\setminus\{(0,0)\}\right\},$$

which is called *the cost of the control at time* T (because it measures the minimal energy needed to bring an initial condition to $(0,0)$). Using a standard duality argument, it can be showed that (3.73) is null controllable if and only if (3.70) is observable, and in this case the cost of the control is

$$\|\Gamma_\omega\|^2 = c(T,a,\omega)^{-1},$$

6. In particular, one has

$$E_a(0) = \int_0^L \left(\varphi_1(x)^2 + \varphi_0'(x)^2 + a(x)\varphi_0(x)^2\right) dx.$$

with $c(T,a,\omega)^{-1}$ the optimal constant in the observability inequality (3.71), defined by (3.72).

The dependence of $c(T,a,\omega)^{-1}$ with respect to different parameters (the observability time T, the potential a, the observability set ω) has been studied by many authors (see [99], where an application to the controllability of semilinear wave equations is given, [32] for some results in the multi-dimensional case obtained thanks to Carleman estimates and [46] for precise lower bounds obtained through different methods) but its exact behavior is not known.

In the following result, one provides several estimates of $c(T,a,\omega)$ (and then $\|\Gamma_\omega\|$) and constitutes another justification of the interest of the problems introduced in Section 3.1.2, in particular of the issue of obtaining a lower bound estimate of the quantity $J(a,\omega)$.

Theorem 5. *Let $L>0$ and let a be a nonnegative function in $L^\infty(0,L)$.*

i There holds

$$c(T,a,\omega)\sim \frac{T}{2}J(a,\omega) \qquad \text{as} \qquad T\to+\infty.$$

ii Let $a\in\mathcal{A}_M(0,L)$ with $M<3\pi^2/L^2$ and define $T(M)=\frac{2\pi}{\gamma_M}$ with $\gamma_M=\frac{\frac{3\pi^2}{L^2}-M}{\frac{2\pi}{L}+\sqrt{\frac{\pi^2}{L^2}+M}}$. For all $T>T(M)$, there holds

$$0<\frac{c_1(T,\gamma_M)}{2}\leqslant\frac{c(T,a,\omega)}{J(a,\omega)}\leqslant\frac{T}{2}$$

with $c_1(T,\gamma_M)=\frac{2}{\pi}\left(T-\frac{4\pi^2}{\gamma_M^2 T}\right)$.

The proof of this theorem is postponed to Section 3.4.4. Combining Theorem 5 with Theorem 3, we have the following result

Corollary 3. *Let $a\in\mathcal{A}_M(0,L)$ with $M\leqslant\pi^2/L^2$ and define $T(M)=\frac{2\pi}{\gamma_M}$ with $\gamma_M=\frac{\frac{3\pi^2}{L^2}-M}{\frac{2\pi}{L}+\sqrt{\frac{\pi^2}{L^2}+M}}$. For all $T>T(M)$, there holds*

$$\frac{7\sqrt{3}}{16L^3}(3-2\sqrt{2})c_1(T,\gamma_M)\min(|\omega|,\underline{r}_2)^3\leqslant c(T,a,\omega),$$

with $\underline{r}_2=\frac{\sqrt{5}}{5}+\frac{\sqrt{10}}{10}$ and $c_1(T,\gamma_M)=\frac{2}{\pi}\left(T-\frac{4\pi^2}{\gamma_M^2 T}\right)$.

3.3.1.2 Randomized observability constant

Let us provide another interpretation of the quantity $J(a,\omega)$. It corresponds also to an averaged version of the observability constant $c(T,a,\omega)$ defined by (3.72), over random initial data. More precisely, let $(\sigma^\nu_{1,j})_{j\in\mathbb{N}^*}$ and $(\sigma^\nu_{2,j})_{j\in\mathbb{N}^*}$ be two sequences of independent Bernoulli random variables on a probability space $(\mathcal{X},\mathcal{A},\mathbb{P})$, satisfying

$$\mathbb{P}(\sigma^\nu_{1,j}=\pm1)=\mathbb{P}(\sigma^\nu_{2,j}=\pm1)=\frac{1}{2}\qquad\text{and}\qquad\mathbb{E}(\sigma^\nu_{1,j}\sigma^\nu_{2,k})=0,$$

for every j and k in $\mathbb{N}^*$ and every $\nu\in\mathcal{X}$. Here, the notation $\mathbb{E}$ stands for the expectation over the space $\mathcal{X}$ with respect to the probability measure $\mathbb{P}$.

Then, $J(a,\omega)$ is the largest constant C for which the inequality

$$C \int_0^L \left(\varphi_1(x)^2 + \varphi_0'(x)^2 + a(x)\varphi_0(x)^2\right)\, dx \leqslant \mathbb{E}\left(\int_0^T \int_\omega |\partial_t \varphi^\nu(t,x)|^2 \, dx\, dt\right), \tag{3.74}$$

holds for all $(\varphi_0, \varphi_1) \in H_0^1(0,L;\mathbb{C}) \times L^2(0,L;\mathbb{C})$, where φ^ν is defined by

$$\varphi^\nu(t,x) = \sum_{j=1}^{+\infty} \left(\sigma_{1,j}^\nu \alpha_j e^{i\lambda_{a,j} t} + \sigma_{2,j}^\nu \beta_j e^{-i\lambda_{a,j} t}\right) e_{a,j}(x),$$

with

$$\alpha_j = \frac{1}{2}\left(\int_0^L \varphi_0(x) e_{a,j}(x)\, dx - \frac{i}{\lambda_{a,j}} \int_0^L \varphi_1(x) e_{a,j}(x)\, dx\right),$$
$$\beta_j = \frac{1}{2}\left(\int_0^L \varphi_0(x) e_{a,j}(x)\, dx + \frac{i}{\lambda_{a,j}} \int_0^L \varphi_1(x) e_{a,j}(x)\, dx\right),$$

for every $j \in \mathbb{N}^*$.

In other words, φ^ν denotes the solution of the wave equation (3.70) with the random initial data $\varphi_0^\nu(\cdot)$ and $\varphi_1^\nu(\cdot)$ determined by their Fourier coefficients $\alpha_j^\nu = \sigma_{1,j}^\nu \alpha_j$ and $\beta_j^\nu = \sigma_{2,j}^\nu \beta_j$.

In this context, the quantity $J(a,\omega)$ is called *randomized observability constant* and we refer to [86, Section 2.3] and [88, Section 2.1] for further explanations on its use in inverse problems. Moreover, a deterministic interpretation of this quantity is provided in [89].

3.3.1.3 Decay rate for a damped wave equation

From the estimates of the observability constant $c(T,a,\omega)$, we can also deduce estimates of the rate at which energy decays in a damped string. Consider the damped wave equation on $(0,\pi)$ with Dirichlet boundary conditions

$$\begin{cases} \partial_{tt} y(t,x) - \partial_{xx} y(t,x) + a(x) y(t,x) + 2k\chi_\omega(x)\partial_t y(t,x) = 0, & (t,x) \in (0,T)\times(0,L), \\ y(t,0) = y(t,\pi) = 0, & t \in [0,T], \\ (y(0,x), \partial_t y(0,x)) = (y^0(x), y^1(x)), & x \in (0,L), \end{cases} \tag{3.75}$$

with $k > 0$. Recall that for all initial data $(y_0, y_1) \in H_0^1(0,\pi) \times L^2(0,\pi)$, the problem (3.75) is well posed and its solution y belongs to $C^0(0,T; H_0^1(0,\pi)) \cap C^1(0,T; L^2(0,\pi))$.

The energy associated to System (3.75) is defined by

$$E_{a,\omega}(t) = \int_0^\pi \left(\partial_t y(t,x)^2 + \partial_x y(t,x)^2 + a(x) y(t,x)^2\right)\, dx.$$

According to Theorem 5 and to [46, Section 3.3], by using the same notations as in the statement of Theorem 5, if ω is a measurable subset of $(0,\pi)$, $a \in \mathcal{A}_M(0,\pi)$ with $M < 3\pi^2/L^2$, there holds for every $(y_0, y_1) \in H_0^1(0,\pi) \times L^2(0,\pi)$ and $t \geqslant 2T(M)$,

$$E_{a,\omega}(t) \leqslant E_{a,\omega}(0) e^{-\delta(a,\omega)t},$$

with

$$\ln\left(\frac{1+(1+T(M)^2)c_1(T,\gamma_M)J(a,\omega)}{(1+T(M)^2)c_1(T,\gamma_M)J(a,\omega)}\right) \leqslant 2T(M)\delta(a,\omega) \leqslant \ln\left(\frac{1+(1+T(M)^2)c_2(T,\gamma_M)J(a,\omega)}{(1+T(M)^2)c_2(T,\gamma_M)J(a,\omega)}\right),$$

where $c_1(T,\gamma_M) = \frac{2}{\pi}\left(T - \frac{4\pi^2}{\gamma_M^2 T}\right)$ and $c_2(T,\gamma_M) = \frac{10T}{\pi}$.

3.3.2 Numerical investigations

In the whole section, we will consider that $L = \pi$ according to Lemma 8, and two given numbers $r \in (0,1)$ and $M \in (0,1]$.

3.3.2.1 The toy case $j_0 = 1$

In the case "$j_0 = 1$", according to Theorem 3, there exist at most two switching points in $(0,\pi)$ denoted o_1 and o_2 such that

$$0 \leqslant o_1 \leqslant o_2 \leqslant \pi \qquad \text{and} \qquad a(x) = M\chi_{(0,o_1)\cup(o_2,\pi)}. \tag{3.76}$$

Therefore, the issue of determining the optimal potential $a(\cdot)$ comes to minimize the function $(0,\pi)^2 \ni (o_1,o_2) \mapsto \inf_{\substack{\omega\subset(0,\pi)\\ \text{s.t. } |\omega|=r\pi}} \int_\omega e_{a,1}(x)^2\,dx$, where $a(\cdot)$ is given by (3.76). Fixing $\tau_a = \sqrt{\lambda_{a,1}^2 - M}$, one computes

$$e_{a,1}(x) = \begin{cases} \sin(\tau_a x) & x\in(0,o_1),\\ \sin(\tau_a o_1)\cos(\lambda_{a,1}(x-o_1)) + \frac{\tau_a}{\lambda_{a,1}}\cos(\tau_a o_1)\sin(\lambda_{a,1}(x-o_1)) & x\in(o_1,o_2),\\ \frac{\sin(\tau_a o_1)\cos(\lambda_{a,1}(o_2-o_1)) + \frac{\tau_a}{\lambda_{a,1}}\cos(\tau_a o_1)\sin(\lambda_{a,1}(o_2-o_1))}{\sin(\tau_a(\pi-o_2))}\sin(\tau_a(\pi-x)) & x\in(o_2,\pi), \end{cases}$$

up to a multiplicative normalization constant, where the eigenvalue $\lambda_{a,1}$ solves the transcendental equation

$$\lambda_{a,1}\tau_a\frac{\tan(\tau_a(\pi-o_2+o_1))}{\tan(\lambda_{a,1}(o_2-o_1))} - \tau_a^2 = M\frac{\sin(\tau_a(\pi-o_2))\sin(\tau_a o_1)}{\cos(\tau_a(\pi-o_2+o_1))}.$$

This last equation is solved numerically by using a Newton method. Since $M \in (0,1]$, the eigenfunction $e_{a,1}$ is concave on $(0,\pi)$. As a consequence, the optimal set ω, as level set of the function $e_{a,1}^2$, writes $\omega = (0,\alpha)\cup(\beta,\pi)$ with $\alpha < \beta$, according to Proposition 5. In that case, we determined α and β with the help of a Newton method, using that $\beta = (1-r)\pi + \alpha$ and $e_{a,1}(\alpha)^2 = e_{a,1}(\beta)^2$.

These considerations allow to rewrite the extremal problem $(\mathcal{P}_{L,r,M})$ as a two-dimensional optimization problem, that we solved numerically by using a Nelder-Mead simplex search method on a standard desktop machine.

3.3.2.2 Numerical solving of $(\mathcal{P}_{j,L,r,M})$ for $j \geqslant 2$.

Let $j \in \mathbb{N}\backslash\{0,1\}$ and $M \in (0,1]$. We fix $o_0 = 0$ and $o_{3j} = \pi$. According to Theorem 3, we reduce the solving of the infinite-dimensional problem $(\mathcal{P}_{j,L,r,M})$ to a $(3j-1)$-dimensional one. In other terms, we minimize the function

$$(0,\pi)^{3j-1} \ni o = (o_1,\cdots,o_{3j-1}) \longmapsto \inf_{\substack{\omega\subset(0,\pi)\\ \text{s.t. } |\omega|=r\pi}} \int_\omega e_{a,1}(x)^2\,dx,$$

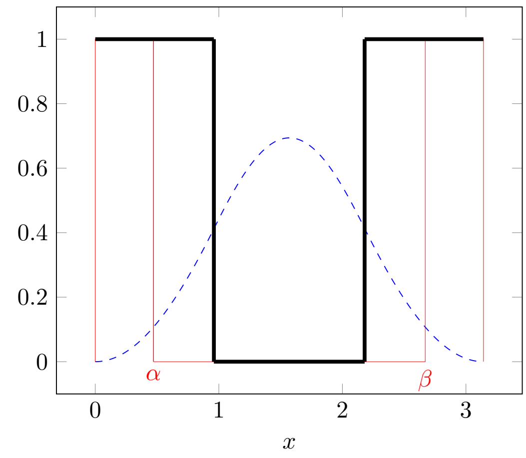

FIGURE 3.7 – $L = \pi$ and $M = 1$. Plots of the optimal set $\omega_1^*(-)$, $a_1^*(\text{-})$ and $e^2_{a_1^*,1}$ $(\dots)$ with respect to the space variable with $r = 0.3$.

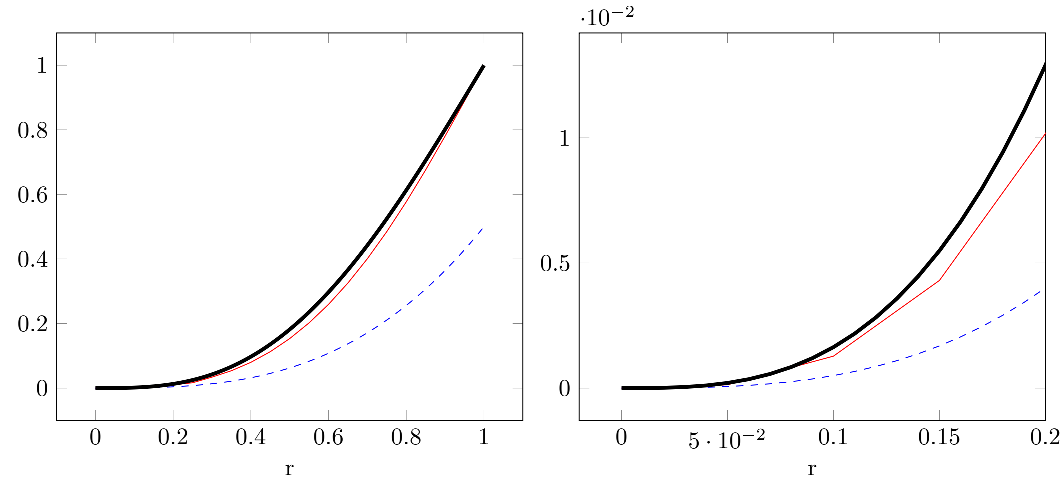

FIGURE 3.8 – $L = \pi$ and $M = 1$. Plot of $r \mapsto m_1(L,r)\,(-)$, $r \mapsto r - \frac{\sin(\pi r)}{\pi}$ (-) and $r \mapsto r^3/2\,(\cdots)$.

where $a(\cdot)$ is a potential function defined on $(0,\pi)$ such that,

$$a(x) = \begin{cases} M & \text{on } (o_i, o_{i+1}), \\ 0 & \text{on } (o_{i+1}, o_{i+2}), \end{cases}$$

for every even integer $i \in \{0, \cdots, 3j-2\}$ and every $x \in (o_i, o_{i+2})$. Notice that, when $3j-2$ is a odd number then $a(x) = M$ on (o_{3j-1}, o_{3j}). Thus, given the switching points $o \in (0,\pi)^{3j-1}$, one computes the eigenfunction $e_{a,j}(\cdot)$ by using a shooting method combined with a Runge-Kutta method. The eigenvalue λ is determined by solving $e_{a,\lambda}(\pi) = 0$ with a Newton method.

According to Proposition 5, the set ω coincides with $\{e^2_{a,j} \leqslant \tau\}$ for some parameter τ chosen in such a way that $|\omega| = r\pi$. We are then driven to find an estimate of τ, which is done by computing the decreasing rearrangement $\left(e^2_{a,j}\right)^*$ of $e^2_{a,j}$ (see, e.g., [50, 59, 90]) and using that

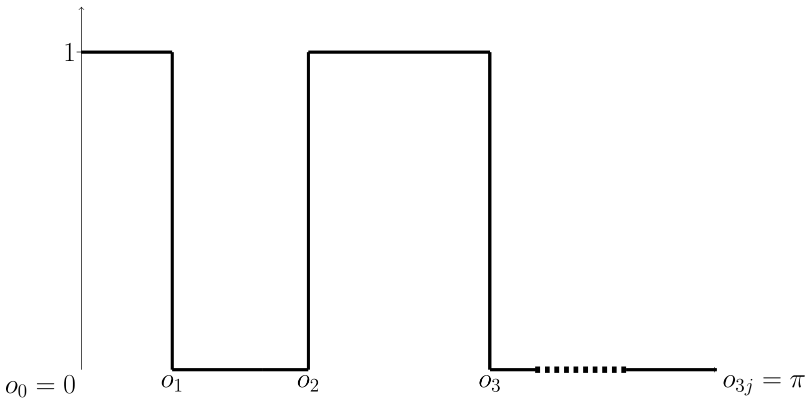

FIGURE 3.9 – Construction of $a(\cdot)$.

$\tau = \left(e^2_{a,j}\right)^* (r\pi)$ (see Figure 3.10) .

These considerations allow to rewrite the cost functional as a function of $(3j-1)$ variables. The resulting finite-dimensional problem is then solved numerically by using a Nelder-Mead simplex search method on a standard desktop machine.

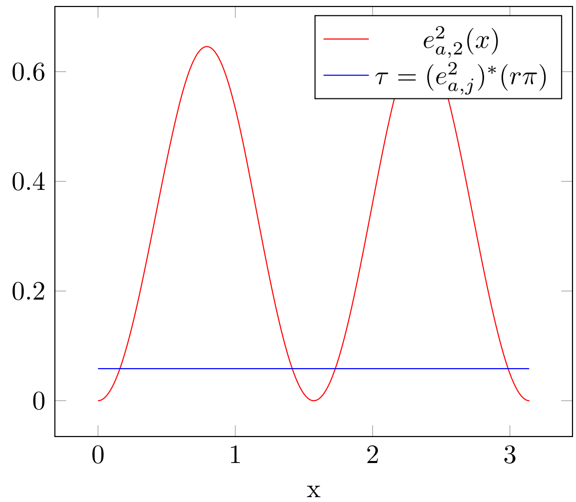

FIGURE 3.10 – $j = 2$, $L = \pi$, $M = 1$ and $r = 0.3$. Graph of $e^2_{a,2}$ and determination of the Lagrange parameter τ [$\left(e^2_{a,j}\right)^*$ is the decreasing rearrangement of $e^2_{a,j}$].

Figures 3.11 and 3.12 illustrate the research of the optimal potential. The parameter r is running over the interval $[0, 1]$. On Figure 3.11, the optimal value of the criterion (w.r.t. r) is compared to the estimate obtained in Theorem 4 for the parameter values $j \in \{2, 3, 4\}$. Recall that the numbers $\underline{m}_j$ are defined in Proposition 4.

On Figure 3.12, the graph of the optimal value with respect to r is plotted for the parameter values $j \in \{1, 2, 6\}$. Notice that the mapping $j \mapsto m_j(L, M, r)$ seems to be increasing, although we did not manage to prove it.

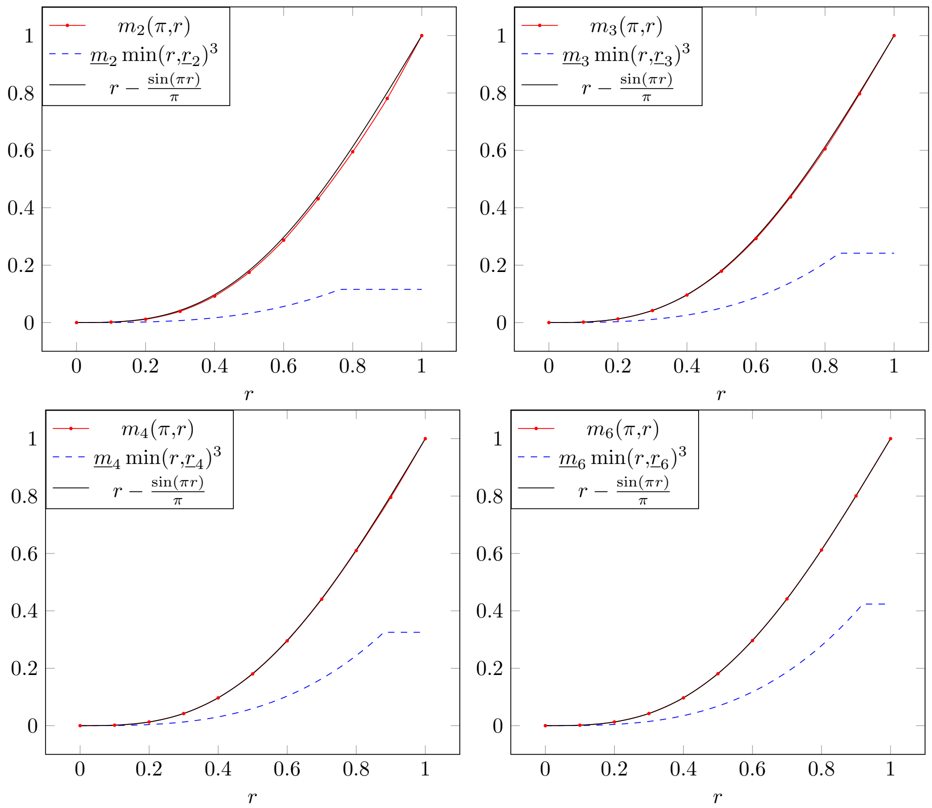

FIGURE 3.11 – $L = \pi$ and $M = 1$. Plots of m_j $(-o-)$, $\underline{m}_j \min(r, \underline{r}_j)^3$ $(\cdots)$ and $r \mapsto r - \frac{\sin(\pi r)}{\pi}$ $(-)$ with respect to r for $j \in \{2, 3, 4\}$.

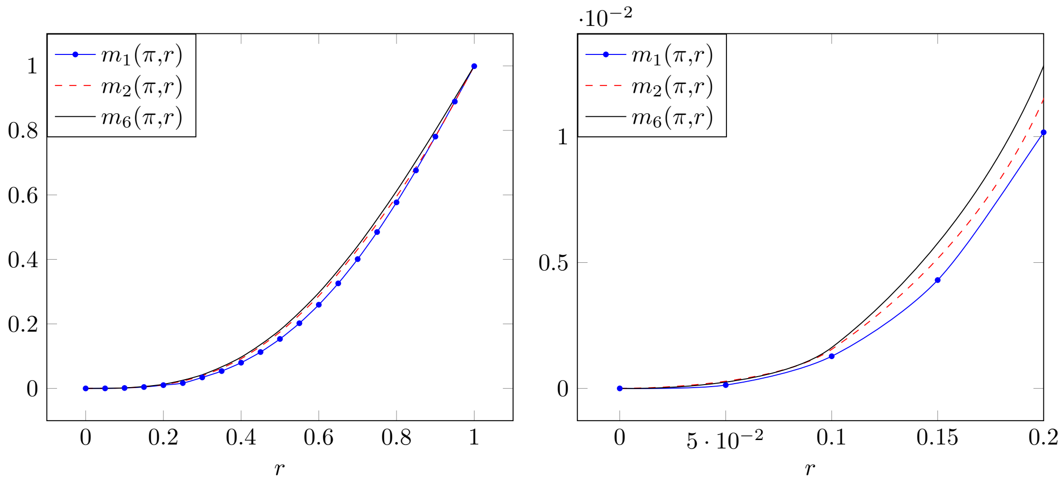

FIGURE 3.12 – $L = \pi$ and $M = 1$. Plots of $m_j(\pi,r)$ for $j = 1$(o), $j = 2$(- -) and $j = 6$ with respect to r.

3.4 Appendix

3.4.1 Proof of Lemma 9

One refers for instance to [49] for a survey of the tools used to prove the continuity of eigenvalues and eigenfunctions with respect to several parameters. Nevertheless, we provide here the main lines of the proof for the sake of completeness.

We first investigate the continuity of the first eigenfunction, in other words, the case "$j = 1$". Let $(a_n)_{n\in\mathbb{N}^*}$ be a sequence of $\mathcal{A}_M(0, L)$ converging for the weak-$\star$ topology of $L^\infty(0, L)$ to a. Then, using that $a_n \leqslant M$ in $(0, L)$ for all $n \in \mathbb{N}^*$ and according to the Courant-Fischer minimax principle, one has

$$\lambda_{a_n,1}^2 = \min_{u\in H_0^1(0,L)\setminus\{0\}} \frac{\int_0^L u'(x)^2 + a_n(x)u(x)^2\,dx}{\int_0^L u(x)^2\,dx} \leqslant \frac{\pi^2}{L^2} + M.$$

It follows that, up to a subsequence, the sequence $(\lambda_{a_n,1}^2)_{n\in\mathbb{N}^*}$ converges to a real nonnegative number denoted λ^2. The variational formulation of the eigenproblem (3.1) writes : find $e_{a_n,1}$ in $H_0^1(0, L)$ such that for every test function $\varphi \in H_0^1(0, L)$, one has

$$\int_0^L e_{a_n,1}'(x)\varphi'(x) + a_n(x)e_{a_n,1}(x)\varphi(x)\,dx = \lambda_{a_n,1}^2 \int_0^L e_{a_n,1}(x)\varphi(x)\,dx. \tag{3.77}$$

Taking as test function $\varphi = e_{a_n,1}$ and using the fact that $\int_0^L e_{a_n,1}(x)^2\,dx = 1$ for every $n \in \mathbb{N}^*$ yields easily that the sequence $(e_{a_n,1})_{n\in\mathbb{N}^*}$ is uniformly bounded in $H_0^1(0, L)$. From Rellich-Kondrachov Theorem, it converges up to a subsequence weakly in $H^1(0, L)$ and strongly in $L^2(0, L)$ to an element f of $H_0^1(0, L)$. Passing into the limit in (3.77) yields that there exists some $j \in \mathbb{N}^*$ such that $f = e_{a,j}$ and $\lambda = \lambda_{a,j}$. Moreover, since

$$\int_0^L e_{a_n,1}'(x)^2\,dx = \lambda_{a_n,1} - \int_0^L a_n(x)e_{a_n,1}(x)^2\,dx,$$

the sequence $(\|e_{a_n,1}\|_{H^1(0,L)})_{n\in\mathbb{N}^*}$ converges to $\|e_{a,j}\|_{H^1(0,L)}$. Combining this fact with the weak convergence of $(e_{a_n,1})_{n\in\mathbb{N}^*}$ to $e_{a,j}$ in $H_0^1(0, L)$ proves that this convergence is in fact strong. Finally, using the embedding $C^0([0, L]) \hookrightarrow H^1(0, L)$ and passing to the limit in the inequality $e_{a_n,1} \geqslant 0$ in $(0, L)$ yields that $e_{a,j}$ is of constant sign. Therefore, there holds necessarily $j = 1$ and the conclusion follows.

To adapt this result to higher orders, it is enough to make the same reasoning on each nodal domain. Without loss of generality, consider the case $j = 2$. Using the same notations as previously, the function $e_{a_n,2}$ has two nodal domains $(0, \xi_n)$ and (ξ_n, L). Moreover, the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges, up to a subsequence to some $\xi^* \in [0, L]$. Applying the previous arguments to $e_{a_n,2}\chi_{(0,\xi_n)}$ and $e_{a_n,2}\chi_{(\xi_n,L)}$ proves that each of these quantities converges strongly in $H^1(0, L)$, respectively to $e_{a,j}\chi_{(0,\xi^*)}$ and $e_{a,j}\chi_{(\xi^*,L)}$. Moreover, one shows as previously that each of these functions is of constant sign and that $\xi^* \in (0, L)$ by passing to the limit in the equality

$$\int_0^L e_{a_n,1}(x)e_{a_n,2}(x)\,dx = 0,$$

which proves that the sign of the function $e_{a,j}$ is non constant. Hence, the function $e_{a,j}$ has two nodal domains, whence the result.

To generalize this result to other eigenfunctions, it is enough to use an induction argument and the L^2 orthogonality of $e_{a,j}$ with $e_{a,j-1}$ that we do not present in details here. The conclusion follows.

3.4.2 Proof of Lemma 11

Let us define $\phi_j = \frac{e_{a_j,j}(\cdot)}{e'_{a_j,j}(0)}$. The function ϕ_j solves the Cauchy system

$$\begin{cases} -\phi_j''(x) + a_j(x)\phi_j(x) = \lambda_{a,j}^2\phi_j(x), & x \in (0,L), \\ \phi_j(0) = 0, \quad \phi_j'(0) = 1. \end{cases}$$

Let us notice that, according to the Courant-Fischer minimax principle, there holds $\lambda_{a_j,j} \geqslant \frac{\pi}{L}$ for every $j \in \mathbb{N}^*$ and $\lim_{j\to+\infty} \lambda_{a_j,j} = +\infty$. According to [57, Chapter 1, Theorem 3] and using a rescaling argument, we infer

$$\phi_j(x) = \frac{\sin(\lambda_{a_j,j}x)}{\lambda_{a_j,j}} + \mathrm{O}\left(\frac{1}{\lambda_{a_j,j}^2}\right).$$

As a consequence, there holds

$$\phi_j^2(x) = \frac{\sin^2(\lambda_{a_j,j}x)}{\lambda_{a_j,j}^2} + \mathrm{O}\left(\frac{1}{\lambda_{a_j,j}^3}\right), \tag{3.78}$$

where the remainder term does not depend on x. Therefore, using the Riemann-Lebesgue lemma, one gets that

$$\int_0^L \phi_j^2(x)dx = \frac{L}{2\lambda_{a,j}^2} + o\left(\frac{1}{\lambda_{a,j}^2}\right), \tag{3.79}$$

and since $e_{a_j,j} = \frac{\phi_j}{\|\phi_j\|_2}$, the combination of (3.78) and (3.79) yields

$$e_{a_j,j}^2(x) = \frac{2}{L}\sin^2(\lambda_{a_j,j}x) + \mathrm{O}\left(\frac{1}{\lambda_{a_j,j}}\right). \tag{3.80}$$

Let $\varphi \in L^1(0,L)$. Using (3.80), one shows that

$$\int_0^L e_{a_j,j}(x)^2\varphi(x)\,dx = \frac{2}{L}\int_0^L \sin^2(\lambda_{a_j,j}x)\varphi(x)\,dx + \mathrm{O}\left(\frac{1}{\lambda_{a_j,j}}\right).$$

By linearizing $\sin^2(\lambda_{a_j,j}x)$ and using the real version of the Riemann-Lebesgue lemma, the expected result follows. ∎

3.4.3 Proofs of Lemmas 15 and 17

The proofs are based on the following Lemma.

Lemma 18. *Let $j \in \mathbb{N}^*$ and $i_0 \in \mathbb{N}^*$ such that $i_0 \leqslant j-1$. Define*

$$g(i_0,j) = \frac{j-i_0}{\sqrt{j^2+1}} + \frac{1}{\sqrt{j^2+1}}\left(\frac{\left(\frac{j^2-2}{j^2}\right)^{\frac{i_0}{2}} - 1}{1-\left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}}\right).$$

There holds

$$g(i_0, j) \geqslant \underline{r}_j := \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right).$$

Démonstration. Let $\gamma : \mathbb{N}\backslash\{0,1\} \ni j \mapsto \left(\frac{j^2}{j^2-2}\right)$. Notice that $\gamma(j) \in (1,2)$ for every $j \in \mathbb{N}\backslash\{0,1\}$. The derivative of g with respect to i_0 writes

$$\partial_{i_0} g(i_0, j) = \frac{-\left(\frac{1}{\gamma(j)}\right)^{\frac{i_0}{2}} \ln(\frac{1}{\gamma(j)}) + 2(\sqrt{\gamma(j)}-1)}{2\sqrt{j^2+1}\left(\sqrt{\gamma(j)}-1\right)} \leqslant 0,$$

and therefore

$$g(i_0, j) \geqslant g(j-1, j) = \frac{\sqrt{\gamma(j)}(\gamma(j)^{\frac{j}{2}}-1)}{\sqrt{j^2+1}(\sqrt{\gamma(j)}-1)\gamma(j)^{\frac{j}{2}}}.$$

Straightforward computations show that

$$\frac{\sqrt{\gamma(j)}(\gamma(j)^{\frac{j}{2}}-1)}{\sqrt{j^2+1}(\sqrt{\gamma(j)}-1)\gamma(j)^{\frac{j}{2}}} = \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right),$$

which concludes the proof. □

Proof of Lemma 15. In the sequel we will use the notations in (3.45), the definition of A_i in (3.52) and the definition of Δ_j in (3.53).

From (3.52) and $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$, we notice that

$$\frac{A_{i+1}}{A_i} = \frac{(1-\eta_i^2)\eta_{i+1}^2}{(1+\eta_{i+1}^2)\eta_i^2} \leqslant 1$$

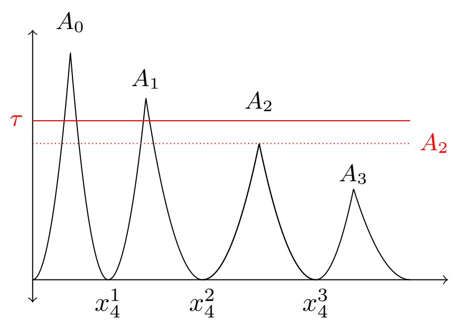

FIGURE 3.13 – $j = 4$, $i_0 = 2$. Graph of Δ_4^2 with respect to x.

Let $i_0 \in \{1, \cdots, j-1\}$ such that for every $i \leqslant i_0 - 1$, $\tau < A_i$ and for every $i \geqslant i_0$, $\tau \geqslant A_i$ (see Figure 3.13 with $j = 4$ and $i_0 = 2$). By definition of $\hat{\omega}$, one has

$$|\hat{\omega}| = |\{\triangle_j(x)^2 < \tau\}| \geqslant |\hat{\omega}_{i_0}|$$

with $\hat{\omega}_{i_0} = \{\triangle_j(\cdot)^2 \leqslant A_{i_0}\}$. Let us denote by $\bar{r}_{i_0} \in (0,1)$ the real number defined by $\bar{r}_{i_0} = \frac{|\{\triangle_j(\cdot)^2 \leqslant A_{i_0}\}|}{L}$. Note that there holds obviously $r \geqslant \bar{r}_{i_0}$.

Let us now find a lower bound estimate of $\bar{r}_{i_0}$. Let $i \in \{0, \cdots, j-1\}$ and let $\omega_i^{i_0} = \hat{\omega}_{i_0} \cap (x_j^i, x_j^{i+1})$. There exist $\alpha_i^{i_0}$, $\beta_i^{i_0}$ and $r_i \in [0,1]$ such that $\omega_i^{i_0} = (x_j^i, \alpha_i^{i_0}) \cup (\beta_i^{i_0}, x_j^{i+1})$, $|\omega_i^{i_0}| = r_i(x_j^{i+1} - x_j^i)$, and therefore

$$\sum_{i=0}^{j-1} r_i(x_j^{i+1} - x_j^i) = \bar{r}_{i_0} L.$$

By definition of $\hat{\omega}_{i_0}$ one has $\triangle_j(\alpha_i^{i_0}) = \triangle_j(\beta_i^{i_0}) = \sqrt{A_{i_0}}$, for every $i \leqslant i_0 - 1$ and $\hat{\omega}_i^{i_0} = (x_j^i, x_j^{i+1})$ for every $i \geqslant i_0$. Consequently there holds

$$\alpha_i^{i_0} = \sqrt{\frac{A_{i_0}}{A_i}}(x_{max}^i - x_j^i) + x_j^i \quad , \quad \beta_i^{i_0} = x_j^{i+1} - \sqrt{\frac{A_{i_0}}{A_i}}(x_j^{i+1} - x_{max}^i).$$

for all $i \leqslant i_0 - 1$. As a result, one gets

$$\begin{aligned} \bar{r}_{i_0} L &= \textstyle\sum_{i=0}^{i_0-1} (\alpha_i^{i_0} - x_j^i + x_j^{i+1} - \beta_i^{i_0}) + (L - x_{i_0}), \\ &= \textstyle\sum_{i=0}^{i_0-1} \sqrt{\frac{A_{i_0}}{A_i}}(x_j^{i+1} - x_j^i) + (L - x_{i_0}). \end{aligned}$$

Dividing by L, we have

$$\bar{r}_{i_0} = \sum_{i=0}^{i_0-1} \sqrt{\frac{A_{i_0}}{A_i}} \eta_i + \sum_{i=i_0}^{j-1} \eta_i, \tag{3.81}$$

where the numbers η_i are defined by (3.45).

This last expression also rewrites

$$\bar{r}_{i_0} = \eta_{i_0} \sum_{i=0}^{i_0-1} \sqrt{\prod_{k=i+1}^{i_0} \left(\frac{1-\eta_{k-1}^2}{1+\eta_k^2}\right)} + \sum_{i=i_0}^{j-1} \eta_i,$$

in terms of the real numbers η_i. Since $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$ for every $j \in \mathbb{N}^*$ and $i \in \{0, \cdots, j-1\}$, one infers that

$$\begin{aligned} \bar{r}_{i_0} &\geqslant \eta_{i_0} \textstyle\sum_{i=0}^{i_0-1} \left(\frac{j^2-2}{j^2}\right)^{\frac{i_0-i}{2}} + \sum_{i=i_0}^{j-1} \eta_i, \\ &\geqslant g(i_0, j), \end{aligned}$$

with

$$g : (i_0, j) \mapsto \frac{j - i_0}{\sqrt{j^2+1}} + \frac{1}{\sqrt{j^2+1}} \left(\frac{\left(\frac{j^2-2}{j^2}\right)^{\frac{i_0}{2}} - 1}{1 - \left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}} \right)$$

By Lemma 18, we conclude that for every $i_0 \geqslant 1$, $r \geqslant \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right)$. As a result, if $r < \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right)$, one has $\tau \leqslant A_i$ for every $i \in \{0, \cdots, j-1\}$.

Proof of Lemma 17. In the sequel we will use the notations in (3.45) and the definition of Δ_j in (3.53) where A_1 is defined in (3.67), A_0 is defined in (3.68) and for all $i \in \{2, \cdots, j-1\}$ A_i is defined in (3.69) .

Let $i_0 \in \{1, \cdots, j-1\}$ and let us introduce the sets

$$I_k = \{i \in \{0, \cdots, j-1\} \text{ such that } A_i > A_k\} \quad \text{and} \quad J_k = \{i \in \{0, \cdots, j-1\} \text{ such that } A_i \leqslant A_k\}.$$

From (3.67), (3.68), (3.69) and $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$, we notice that

$$\frac{A_{i+1}}{A_i} = \frac{(1-\eta_i^2)\eta_{i+1}^2}{(1+\eta_{i+1}^2)\eta_i^2} \leqslant 1, \quad \text{for all} \quad i \in \{1, \cdots, j-1\} \quad \text{and} \quad \frac{A_0}{A_1} \leqslant 1. \tag{3.82}$$

Case 1 : $0 \in J_{i_0}$ *(see Figure 3.14).*

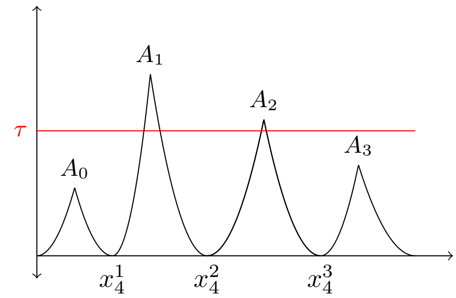

FIGURE 3.14 – $j = 4$, $i_0 = 2$. Graph of Δ_4^2 with respect to x.

Thanks to (3.82), one has $J_{i_0} = \{0, i_0+1, \cdots, j-1\}$. By mimicking the proof of Lemma 15 (notably the equality (3.81)), we have

$$\begin{aligned} r &= \sum_{i=1}^{i_0} \sqrt{\frac{A_{i_0+1}}{A_i}}\eta_i + \sum_{i \in J_{i_0}} \eta_i, \\ &\geqslant \eta_{i_0} \sum_{i=1}^{i_0} \sqrt{\prod_{k=i+1}^{i_0+1} \left(\frac{1-\eta_{k-1}^2}{1+\eta_k^2}\right)} + \sum_{i=i_0+1}^{j-1} \eta_i + \eta_0, \\ &\geqslant \eta_{i_0} \sum_{i=1}^{i_0} \left(\frac{j^2-2}{j^2}\right)^{\frac{i_0+1-i}{2}} + \sum_{i=i_0+1}^{j-1} \eta_i + \eta_0, \\ &\geqslant g(i_0, j), \end{aligned}$$

with

$$g : (i_0, j) \mapsto \frac{j - i_0}{\sqrt{j^2+1}} + \frac{1}{\sqrt{j^2+1}} \left(\frac{\left(\frac{j^2-2}{j^2}\right)^{\frac{i_0}{2}} - 1}{1 - \left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}} \right),$$

which leads to the desired conclusion using Lemma 18.

Case 2 : $0 \in I_{i_0}$ *(see Figure 3.15).*

Thanks to (3.82), one has $J_{i_0} = \{i_0, \cdots, j-1\}$. By mimicking the proof of Lemma 15, we

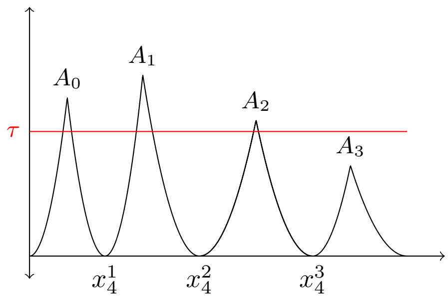

FIGURE 3.15 – $j = 4$, $i_0 = 3$. Graph of Δ_4^2 with respect to x.

have

$$r_{i_0} = \sqrt{\frac{A_{i_0}}{A_0}}\eta_0 + \sum_{i=1}^{i_0-1} \sqrt{\frac{A_{i_0}}{A_i}}\eta_i + \sum_{i=i_0}^{j-1} \eta_i$$

Thanks to (3.68), (3.69) and $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$,

$$\sqrt{\frac{A_{i_0}}{A_0}}\eta_0 \geqslant \eta_{i_0}\sqrt{\frac{(j^2+2)(j^2-1)}{j^2(j^2+1)}}\left(\frac{j^2}{j^2-2}\right)^{\frac{i_0-2}{2}}$$

and

$$\sqrt{\frac{A_{i_0}}{A_i}}\eta_i \geqslant \eta_{i_0}\left(\frac{j^2-2}{j^2}\right)^{\frac{i_0-i}{2}}.$$

Since the inequality $\frac{(j^2+2)(j^2-1)}{j^2(j^2+1)} \geqslant \left(\frac{j^2}{j^2-2}\right)^2$ is true for every $j \geqslant 1$, we thus infer

$$\begin{aligned} \bar{r}_{i_0} &\geqslant \eta_{i_0} \sum_{i=0}^{i_0-1} \left(\frac{j^2-2}{j^2}\right)^{\frac{i_0-i}{2}} + \sum_{i=i_0}^{j-1} \eta_i, \\ &\geqslant g(i_0, j). \end{aligned}$$

By using Lemma 18, we conclude the proof. ■

3.4.4 Proof of Theorem 5

Before proving this theorem, let us recall some basic facts on Ingham's inequality (see [53]), an inequality for nonharmonic Fourier series much used in control theory.

Proposition 8. *For every $\gamma > 0$ and every $T > \frac{2\pi}{\gamma}$, there exist two positive constants $C_1(T,\gamma)$ and $C_2(T,\gamma)$ such that for every sequence of real numbers $(\mu_n)_{n\in\mathbb{N}^*}$ satisfying*

$$\forall n \in \mathbb{N}^* \quad |\mu_{n+1} - \mu_n| \geqslant \gamma, \tag{3.83}$$

there holds

$$C_1(T,\gamma)\sum_{n\in\mathbb{Z}^*} |a_n|^2 \leqslant \int_0^T \left|\sum_{n\in\mathbb{Z}^*} a_n e^{i\mu_n t}\right|^2 dt \leqslant C_2(T,\gamma)\sum_{n\in\mathbb{Z}^*} |a_n|^2, \tag{3.84}$$

for every $(a_n)_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{C})$.

Denoting by $C_1(T,\gamma)$ and $C_2(T,\gamma)$ the optimal constants in (3.84), several explicit estimates of these constants are provided in [53]. For example, it is proved in the article cited above that

$$C_1(T,\gamma) \geqslant 2\left(\frac{T}{\pi} - \frac{4\pi}{\gamma^2 T}\right) \quad \text{and} \quad C_2(T,\gamma) \leqslant \frac{10T}{\pi}.$$

The idea to use Ingham inequalities in control theory is a long story (see for instance [14, 43, 54, 55, 62]).

Notice that, up to our knowledge, the best constants in [53] are not known. In the particular case where $\mu_n = \pi n/L$ for every $n \in \mathbb{N}^*$, one shows easily that for every $T > 2L$, $C_1(T,\gamma) = 2\pi\left\lfloor \frac{T}{2\pi}\right\rfloor$ and $C_2(T,\gamma) = C_1(T,\gamma)+1$, the bracket notation standing for the integer floor.

The following result on the asymptotic as $T \to +\infty$ of optimal constants Ingham's inequalities will be a crucial tool to prove Theorem 5.

Proposition 9. *Assume that the sequence* $(\mu_n)_{n\in\mathbb{N}^*}$ *satisfies* (3.83)*. Then, there holds*

$$\lim_{T\to+\infty} \frac{C_1(T,\gamma)}{T} = \lim_{T\to+\infty} \frac{C_2(T,\gamma)}{T} = 1.$$

Proof of Proposition 9. Let $(a_n)_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{C})$ be such that $\|a\|_{\ell^2} = 1$. Introduce the quantity

$$Q_T(a,\mu) = \int_0^T \left|\sum_{n\in\mathbb{Z}^*} a_n \mathrm{e}^{i\mu_n t}\right|^2 dt.$$

We write

$$\begin{aligned} Q_T(a,\mu) &= \int_0^T \sum_{n\in\mathbb{Z}^*} |a_n|^2\, dt + \int_0^T \sum_{n\neq m} a_n \bar{a}_m e^{i(\mu_n-\mu_m)t}\, dt \\ &= T - i\sum_{n\neq m} \frac{a_n\bar{a}_m(e^{i(\mu_n-\mu_m)T}-1)}{\mu_n-\mu_m} \\ &= T - i\sum_{n\neq m}\frac{b_n\bar{b}_m}{\mu_n-\mu_m} + i\sum_{n\neq m}\frac{a_n\bar{a}_m}{\mu_n-\mu_m}, \end{aligned}$$

with $b_n = a_n e^{i\mu_n T}$ for every $n \in \mathbb{Z}^*$.

According to [80, Theorem 2], one has

$$\left|\sum_{n\neq m}\frac{b_n\bar{b}_m}{\mu_n-\mu_m}\right| \leqslant \frac{\pi}{\gamma}\|a\|_{\ell^2}^2 \quad \text{and} \quad \left|\sum_{n\neq m}\frac{a_n\bar{a}_m}{\mu_n-\mu_m}\right| \leqslant \frac{\pi}{\gamma}\|a\|_{\ell^2}^2,$$

where $\gamma = \inf_{n\in\mathbb{Z}^*} \mu_{n+1}-\mu_n$. Then, it follows that

$$1-\frac{\pi}{\gamma T} \leqslant \frac{1}{T}\inf_{a\in\ell^2 \| \|a\|_{\ell^2}=1} Q_T(a,\mu) \leqslant \frac{1}{T}\sup_{a\in\ell^2 \| \|a\|_{\ell^2}=1} Q_T(a,\mu) \leqslant 1+\frac{\pi}{\gamma T},$$

whence the result. ■

Decomposing the solution φ of (3.70) in the spectral basis $\{e_{a,j}\}_{j\in\mathbb{N}^*}$ allows to write that

$$\varphi(t,x)=\sum_{j=1}^{+\infty}(\alpha_j\cos(\lambda_{a,j}t)+\beta_j\sin(\lambda_{a,j}t))\,e_{a,j}(x), \tag{3.85}$$

where

$$\alpha_j=\int_0^L\varphi_0(x)e_{a,j}(x)\,dx,\qquad \beta_j=\frac{1}{\lambda_{a,j}}\int_0^L\varphi_1(x)e_{a,j}(x)\,dx, \tag{3.86}$$

for every $j\in\mathbb{N}^*$. We are now ready to prove Theorem 5.

Proof of Theorem 5. Introduce the spectral gap

$$\gamma=\inf_{j\in\mathbb{N}^*}\lambda_{a,j+1}-\lambda_{a,j}. \tag{3.87}$$

It is well-known that $\gamma>0$ for every $a\in L^\infty(0,L)$. Let us first prove point (i). Using (3.84), one has for $T\geqslant 2\pi/\gamma$,

$$\begin{aligned}\int_0^T\int_\omega|\partial_t\varphi(t,x)|^2\,dxdt &= \frac{1}{4}\int_0^T\int_\omega\left|\sum_{k\in\mathbb{Z}^*}i\,\mathrm{sgn}(k)\lambda_{a,|k|}\sqrt{\alpha_{|k|}^2+\beta_{|k|}^2}e^{i\,\mathrm{sgn}(k)(\lambda_{a,|k|}t-\theta_{|k|})}e_{a,|k|}(x)\right|^2 dxdt\\ &\geqslant \frac{C_1(T,\gamma)}{4}\sum_{k\in\mathbb{Z}^*}(\alpha_{|k|}^2+\beta_{|k|}^2)\lambda_{a,|k|}^2\int_\omega e_{a,|k|}(x)^2\,dx\\ &= \frac{C_1(T,\gamma)}{2}\sum_{j=1}^{+\infty}(\alpha_j^2+\beta_j^2)\lambda_{a,j}^2\int_\omega e_{a,j}(x)^2\,dx,\end{aligned} \tag{3.88}$$

where $(\theta_j)_{j\in\mathbb{N}^*}$ denotes the sequence defined by $e^{i\theta_j}=\frac{\alpha_j+i\beta_j}{\sqrt{\alpha_j^2+\beta_j^2}}$ for every $j\in\mathbb{N}^*$. Combining the energy identity

$$\int_0^L\left(\varphi_t^2(t,x)+\varphi_x^2(t,x)+a(x)\varphi^2(t,x)\right)\,dx=\sum_{j=1}^{+\infty}\lambda_{a,j}^2(\alpha_j^2+\beta_j^2), \tag{3.89}$$

with (3.72), (3.85), (3.86) and (3.89), one gets

$$c(T,a,\omega)=\inf_{(\lambda_{a,j}\alpha_j,\lambda_{a,j}\beta_j)\in l^2(\mathbb{R})^2}\frac{\int_0^T\int_\omega|\sum_{j=1}^{+\infty}(-\lambda_{a,j}\alpha_j\sin(\lambda_{a,j}t)+\lambda_{a,j}\beta_j\cos(\lambda_{a,j}t))\,e_{a,j}(x)|^2\,dxdt}{\sum_{j=1}^{+\infty}\lambda_{a,j}^2(\alpha_j^2+\beta_j^2)}, \tag{3.90}$$

According to (3.88), it follows that

$$c(T,a,\omega)\geqslant\frac{C_1(T,\gamma)}{2}\inf_{j\in\mathbb{N}^*}\int_\omega e_{a,j}(x)^2\,dx. \tag{3.91}$$

Taking now $\alpha_j=(\delta_{kj})_{k\in\mathbb{N}^*}$ and $\beta_j=(\delta_{k'j})_{k'\in\mathbb{N}^*}$ in (3.90) where δ_{kj} denotes the Kronecker delta, we obtain

$$c(T,a,\omega)\leqslant\frac{T}{2}\inf_{j\in\mathbb{N}^*}\int_\omega e_{a,j}(x)^2\,dx. \tag{3.92}$$

Combining (3.91), (3.92) with the asymptotic of optimal constant $C_1(T,\gamma)$ in Ingham's inequalities stated in Proposition 9 leads to the desired result. Let us now prove point ii. According to (3.89), (3.91) and (3.92), there holds

$$0 < \frac{C_1(T,\gamma)}{2} \leqslant \frac{c(T,a,\omega)}{\inf_{j\in\mathbb{N}^*}\int_\omega e_{a,j}(x)^2\,dx} \leqslant \frac{T}{2} \tag{3.93}$$

with $C_1(T,\gamma) \geqslant 2\left(\frac{T}{\pi} - \frac{4\pi}{\gamma^2 T}\right)$. To conclude, it remains to provide an estimate of the spectral gap γ defined by (3.87).

Lemma 19. *Let $a \in \mathcal{A}_M(0,L)$ with $M \in (0, 3\pi^2/L^2)$. There holds*

$$\lambda_{a,j+1} - \lambda_{a,j} \geqslant \frac{\frac{3\pi^2}{L^2} - M}{\frac{2\pi}{L} + \sqrt{\frac{\pi^2}{L^2} + M}},$$

for every $j \in \mathbb{N}^*$.

Démonstration. The Courant-Fischer minimax principle writes

$$\lambda_{a,j}^2 = \min_{\substack{V\subset H_0^1(0,\pi)\\ \dim V = j}} \max_{u\in V\setminus\{0\}} \frac{\int_0^\pi (u'(x)^2 + a(x)u(x)^2)dx}{\int_0^\pi u(x)^2\,dx}. \tag{3.94}$$

Using that $0 \leqslant a(x) \leqslant M$ for almost every $x \in (0,L)$ yields

$$\frac{j\pi}{L} \leqslant \lambda_{a,j} \leqslant \sqrt{\left(\frac{j\pi}{L}\right)^2 + M},$$

for every $j \in \mathbb{N}^*$. It suffices indeed to compare $\lambda_{a,j}^2$ with the j-th eigenvalue of a Sturm-Liouville operator with constant coefficients. We infer

$$\begin{aligned} \lambda_{a,j+1} - \lambda_{a,j} &\geqslant \frac{(j+1)\pi}{L} - \sqrt{\left(\frac{j\pi}{L}\right)^2 + M}, \\ &= \frac{(1+2j)\frac{\pi^2}{L^2} - M}{(j+1)\frac{\pi}{L} + \sqrt{\left(\frac{j^2\pi^2}{L^2}\right) + M}}. \end{aligned}$$

for every $j \in \mathbb{N}^*$. The sequence $j \mapsto \frac{(1+2j)\frac{\pi^2}{L^2} - M}{(j+1)\frac{\pi}{L} + \sqrt{\left(\frac{j^2\pi^2}{L^2}\right)+M}}$ being increasing, the expected estimate follows. □

Therefore, according to Lemma 19, one has $\gamma \geqslant \gamma_M$. Hence, choosing $\gamma = \gamma_M$ in (3.93) for all $T > T(M) = \frac{2\pi}{\gamma_M}$ yields the expected conclusion. ■

Deuxième partie

Contrôlabilité de quelques systèmes d'équations couplées

Chapitre 4

Introduction

Ces résultats sont extraits de [71] (en collaboration avec Pierre Lissy).

4.1 Mise en place du problème

Soit Ω un ouvert de $\mathbb{R}^N$ avec $N \in \mathbb{N}^*$ et ω un ouvert inclus dans Ω. Nous allons prouver, sous des conditions de type Kalman, un résultat de contrôlabilité pour certains systèmes d'équations avec un nombre réduit de contrôles. Ces équations seront couplées par une matrice constante et le contrôle, à support dans un sous-ensemble ω, agira aussi au travers d'une matrice constante. Tout d'abord, plaçons nous dans le cas où nous avons une équation de Schrödinger avec un contrôle interne v

$$\left\{ \begin{array}{l} \partial_t y = i\Delta y + \mathbb{1}_\omega v, \quad \text{sur } (0,T)\times\Omega, \\ y(0) = y^0 \quad \text{sur } (0,T)\times\partial\Omega, \end{array} \right. \tag{4.1}$$

avec

$$\mathbb{1}_\omega = \left\{ \begin{array}{lll} 1 & \text{on} & \omega, \\ 0 & \text{on} & \Omega\backslash\omega. \end{array} \right.$$

Supposons que l'une des conditions suivantes est vérifiée :

— (ω, T) vérifie GCC[1],

— l'ensemble $\Omega \subset \mathbb{R}^N$ est le produit de N intervalles ouverts,

— l'ensemble $\Omega \subset \mathbb{R}^2$ est le disque unité et $\omega \cap \partial\Omega \neq \emptyset$,

alors, d'après respectivement [66], [62, Proposition 8.8] et [12, Théorème 1.2], pour tout $T > 0$, il existe un contrôle v dans $C^0([0,T];L^2(\Omega))$ tel que $y(T,\cdot) = 0$. Considérons maintenant le cas où nous avons deux équations de Schrödinger couplées avec un contrôle v

$$\left\{ \begin{array}{l} \partial_t y_1 = i\Delta y_1 + a_{11}y_1 + a_{12}y_2 + \mathbb{1}_\omega v, \quad \text{sur } (0,T)\times\Omega, \\ \partial_t y_2 = i\Delta y_2 + a_{21}y_1 + a_{22}y_2 \quad \text{sur } (0,T)\times\Omega, \\ y_1(0) = y_1^0, y_2(0) = y_2^0 \quad \text{sur } (0,T)\times\partial\Omega, \end{array} \right. \tag{4.2}$$

où $a_{ij} \in \mathbb{R}$ pour tout $(i,j) \in \{1,2\}^2$. L'équation (4.2) est **contrôlable à** 0 **en temps** T si et seulement si pour tout (y_1^0, y_2^0) dans $L^2(\Omega)^2$, il existe un contrôle v dans $L^2((0,T),(0,L))$ tel que $y_1(T,\cdot) = y_2(T,\cdot) = 0$. On remarque que le contrôle u agit *indirectement* sur la solution y_2 par l'intermédiaire du terme $a_{21}y_1$. Ainsi, si $a_{21} = 0$ alors le système devient découplé et nous ne

1. Rappelons que (ω, T) satisfait la condition de contrôle géométrique (GCC) dans Ω si chaque rayon de l'optique géométrique qui se propage dans Ω et qui se reflète sur sa frontière entre dans ω en un temps plus petit que T.

pouvons donc plus agir sur y_2. De plus, lorsque $a_{11} = a_{12} = 0$, nous nous ramenons à l'étude de la contrôlabilité de l'équation (4.1). Généralisons le problème (4.2) à n équations et à m contrôles internes. Nous considérons

$$\begin{cases} \partial_t Y = i\Delta Y + AY + \mathbb{1}_\omega BV, & \text{sur } (0,T) \times \Omega, \\ Y(0) = Y^0 & \text{sur } (0,T) \times \partial\Omega, \end{cases} \tag{4.3}$$

avec $V \in L^2(0,T;\Omega)^m$ un contrôle et $(A,B) \in \mathcal{M}_n(\mathbb{R}) \times \mathcal{M}_{n,m}(\mathbb{R})$. Nous allons agir sur ce système avec moins de contrôle que d'équations, c'est à dire en supposant que $m < n$.

Quelles sont les conditions sur A, B, ω, T et Y^0 pour que le contrôle V amène la solution Y de (4.3) à l'état d'équilibre 0 ?

On se posera la même question pour le système d'équations des ondes

$$\begin{cases} \partial_{tt} Y = \Delta Y + AY + \mathbb{1}_\omega BV, & \text{sur } (0,T) \times \Omega, \\ Y(0) = Y^0 & \text{sur } (0,T) \times \partial\Omega. \end{cases}$$

Nous répondrons à cette question dans le cadre plus général suivant : soit $\mathcal{U}$ et $\mathcal{H}$ deux espaces de Hilbert sur $\mathbb{C}$ et soit $\mathcal{C} : \mathcal{U} \to \mathcal{H}$ une application linéaire continue. Considérons

— $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ un opérateur non borné, fermé avec domaine dense qui est supposé être le générateur d'un **groupe** sur $\mathcal{H}$.

— $\mathcal{Q} : \mathcal{D}(\mathcal{Q}) \subset \mathcal{H} \to \mathcal{H}$ un opérateur non borné, fermé avec domaine dense qui est supposé être **autoadjoint et négatif avec résolvante compacte**.

La réversibité en temps des équations associées à $\mathcal{L}$ et $\mathcal{Q}$ est une hypothèse fondamentale pour construire un contrôle régulier lors de la résolution du problème analytique (voir Section 4.4 et [Chapitre 5, Section 5.2]). Pour tout $k \in \mathbb{N}^*$, nous introduisons les opérateurs $L_k : \mathcal{D}(\mathcal{L})^k \subset \mathcal{H}^k \to \mathcal{H}^k$, $Q_k : \mathcal{D}(\mathcal{Q})^k \subset \mathcal{H}^k \to \mathcal{H}^k$ et $C_k : \mathcal{U}^k \to \mathcal{H}^k$ tels que, pour tout $\varphi \in \mathcal{D}(\mathcal{L})^k$, $\phi \in \mathcal{D}(\mathcal{Q})^k$ et $\psi \in \mathcal{U}^k$,

$$L_k(\varphi) = \begin{pmatrix} \mathcal{L}(\varphi_1) \\ \mathcal{L}(\varphi_2) \\ \vdots \\ \mathcal{L}(\varphi_k) \end{pmatrix}, \quad Q_k(\phi) = \begin{pmatrix} \mathcal{Q}(\phi_1) \\ \mathcal{Q}(\phi_2) \\ \vdots \\ \mathcal{Q}(\phi_k) \end{pmatrix} \quad \text{et} \quad C_k(\psi) = \begin{pmatrix} \mathcal{C}(\psi_1) \\ \mathcal{C}(\psi_2) \\ \vdots \\ \mathcal{C}(\psi_k) \end{pmatrix}. \tag{4.4}$$

Nous considérons le système du premier et deuxième ordre à n équations linéaires couplées et à m contrôles

$$\partial_t Y = L_n(Y) + AY + BC_m V \quad \text{dans} \quad (0,T) \times \mathcal{H}^n, \tag{Ord1}$$

$$\partial_{tt} Y = Q_n(Y) + AY + BC_m V \quad \text{dans} \quad (0,T) \times \mathcal{H}^n, \tag{Ord2}$$

avec $(A,B) \in \mathcal{M}_n(\mathbb{R}) \times \mathcal{M}_{n,m}(\mathbb{R})$ des matrices constantes.

Problème : Déterminer des conditions sur

1) La matrice de couplage A et la matrice de contrôle B,
2) Le temps d'observation T et l'application linéaire C_m,
3) L'espace des données initiales Y^0,

pour qu'il existe un contrôle V tel que

$$Y(T,\cdot) = 0.$$

En choisissant $\mathcal{L} = i\Delta$ (resp. $\mathcal{Q} = \Delta$) et $\mathcal{C} = \mathbb{1}_\omega$ dans (Ord1) (resp. dans (Ord2)), nous retrouvons le système (4.3) (resp. (4.7))).

4.2 Résultats antérieurs

Plusieurs auteurs se sont intéressés à la contrôlabilité des systèmes hyperboliques pour montrer, par exemple, grâce à des méthodes de transmutations (voir [92] et [78]), de nouveaux résultats sur des systèmes paraboliques.

Soit $\omega \subset \Omega$ un ouvert. Considérons n équations des ondes et m contrôles sur $L^2(\Omega)$

$$\partial_{tt}Y - \Delta Y = A(x)Y + \chi_\omega BV(t,x), \quad (t,x) \in (0,T) \times \Omega, \tag{4.5}$$

avec $A : \Omega \to \mathcal{M}_n(\mathbb{R})$ et $\chi_\omega : \Omega \to \mathbb{R}$ des fonctions régulières sur Ω. On suppose, de plus, que χ_ω est à support dans ω et $B \in \mathcal{M}_{m,n}(\mathbb{R})$. Un des premiers résultats est obtenu dans [3, 4] en se plaçant dans le cas $A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & a(x) \end{pmatrix}$ et $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. En introduisant une méthode d'énergie à deux niveaux, les auteurs prouvent un résultat de contrôlabilité à 0 en supposant que $\{b \neq 0\}$ et (ω, T) vérifient *GCC* avec un temps T suffisamment grand.

La méthode d'énergie à deux niveaux a été généralisée dans [1] pour des systèmes à n équations et à m contrôles avec une matrice de couplage en cascade. Plus précisément, si $B = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$ et $A(x) = \begin{pmatrix} 0 & c_{21}(x) & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & & \ddots & c_{nn-1}(x) \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$ alors, d'après [1], sous la condition GCC sur (ω, T) et $\{c_{ii-1} \neq 0\}$ pour tout $i \in \{2 \cdots n\}$ et sous des conditions de compatibilités sur $\{c_{ii-1} \neq 0\}$ pour tout $i \in \{2 \cdots n\}$, nous avons contrôlabilité à 0 pour un temps suffisamment grand et pour des données initiales vivant dans l'espace naturel $\prod_{i=1}^n \mathcal{D}(\Delta^{(n-i-1)/2}) \times \prod_{i=1}^n \mathcal{D}(\Delta^{(n-i)/2})$.

Les méthodes de transmutation (voir [92] et [79, Section 5.3]) permettent de transférer les résultats sur la contrôlabilité d'un système hyperbolique à un système parabolique de la forme

$$\partial_t Y - \Delta Y = A(x)Y + \chi_\omega BV(t,x), \quad (t,x) \in (0,T) \times \Omega.$$

Ainsi les résultats dans [1, 3, 4] n'imposent pas la condition $\{b \neq 0\} \cap \omega \neq \emptyset$ (ou $\{c_{ii-1} \neq 0\} \cap \omega \neq \emptyset$) pour tout $i \in \{2 \cdots n\}$ nécessaire dans [98, 7, 38] pour appliquer des inégalités de Carleman. Cependant, la condition GCC pour les systèmes paraboliques est inappropriée (voir par exemple [9]).

Par des méthodes de transmutation (voir [83] et [78]), tous les résultats de contrôlabilité pour le système (4.5) se transfère à un système d'équations de Schrödinger de la forme

$$\partial_t Y - i\Delta Y = A(x)Y + \chi_\omega BV(t,x), \quad (t,x) \in (0,T) \times \Omega. \tag{4.6}$$

Cependant la condition GCC sur (ω, T) est une condition suffisante mais non nécessaire de contrôlabilité pour l'équation de Schrödinger et l'espace des données initiales $\prod_{i=1}^n \mathcal{D}(\Delta^{(n-i)/2})$ n'est pas un espace naturel à cause du caractère non régularisant de l'opérateur de Schrödinger.

Dans [76], les auteurs considèrent un système à deux équations couplées avec une matrice

$A(x) = \begin{pmatrix} 0 & 0 \\ \mathbb{1}_{\mathcal{O}}(x) & 0 \end{pmatrix}$ et à un contrôle agissant au travers d'un vecteur $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Ils imposent que $\mathcal{O} \cap \omega \neq \emptyset$ et des conditions techniques sur ω et $\mathcal{O}$ pour prouver une inégalité de Carleman. Dans ce cas, (ω, T) ne vérifie pas GCC et donc ce résultat de contrôlabilité ne peut pas découler du système (4.5). Pour le même système que dans [76] et sans imposer non plus une condition GCC sur (ω, T), nous pouvons citer aussi [91] où Ω est un tore $\mathbb{T}^N = \mathbb{R}/(2\pi\mathbb{Z})^N$. La preuve est basée sur la périodicité du groupe d'isométries $(S(t))_{t\in\mathbb{R}}$.

4.3 Résultats obtenus

4.3.1 Théorèmes principaux

Énonçons la condition nécessaire et suffisante de type Kalman pour que le système d'équations d'ordre deux en temps (Ord2) soit contrôlable à 0.

Théorème 3. *Sous des conditions techniques "de contrôlabilité, de régularité et de localité" sur $\mathcal{Q}$, pour tout $T > T^*$, il existe un contrôle V dans $C^0([0,T] \times \mathcal{U}^m)$ tel que la solution Y de* (Ord2) *correspondant à la condition initiale $Y(0) = Y^0$ dans $\mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \times \mathcal{D}(\mathcal{Q}^{n-1})^n$ vérifie $Y(T,\cdot) = 0$ si et seulement si rang$(B|AB|A^2B|\cdots|A^{n-1}B) = n$.*

Nous renvoyons au [Chapitre 5, Assumptions] pour la formulation exacte des conditions sur Q et T^*. Dans la Section 4.3.2, nous allons mettre en avant les conséquences de ces conditions dans le cas d'un opérateur des ondes (**H1-ondes** et **H2-ondes**).

Le théorème suivant traite du cas des systèmes d'ordre un en temps. Ce résultat caractérise exactement les conditions initiales qui sont contrôlables et donc ne peut pas découler du Théorème 3 par des méthodes de transmutation (voir [83] et [77]). Cette caractérisation était seulement connue pour le cas de la dimension finie et pour les systèmes paraboliques (voir [5, Théorème 1.5]).

Théorème 4. *Sous des conditions techniques "de contrôlabilité, de régularité et de localité" sur $\mathcal{L}$, pour tout $T > 0$, il existe un contrôle V dans $C^0([0,T] \times \mathcal{U}^m)$ tel que la solution Y de* (Ord1) *correspondant à la condition initiale $Y(0) = Y^0$ dans $(\mathcal{D}(\mathcal{L})^{n-1})^n$ satisfasse $Y(T,\cdot) = 0$ si et seulement si $Y^0 \in (B|AB|A^2B|\cdots|A^{n-1}B)(\mathcal{H}^{nm})$.*

De même, les conditions sur $\mathcal{L}$ seront explicitées dans le cas d'un opérateur de Schrödinger (voir Section 4.3.2). Si nous transformons le système (Ord2) en un système du premier ordre, nous ne pourrons pas trouver une matrice A telle que le système (Ord2) s'écrit comme le système (Ord1). La matrice A doit agir simultanément sur Y et Y_t, ce qui n'est pas le cas ici (voir [Chapitre 5, (5.39)]). Ainsi, le Théorème 3 n'est pas un cas particulier du Théorème 4.

4.3.2 Applications

En appliquant le Théorème 3, nous prouvons que si (ω, T) vérifie GCC alors le système

$$\begin{cases} \partial_{tt} Y = \Delta Y + AY + \mathbb{1}_\omega BV & \text{dans} \quad (0,T) \times L^2(\Omega)^n, \\ Y(0) = Y^0, \\ \partial_t Y(0) = Y^1, \end{cases} \tag{4.7}$$

avec $(A, B) \in \mathcal{M}_n(\mathbb{R}) \times \mathcal{M}_{n,m}(\mathbb{R})$ est contrôlable à 0 au temps T pour des données initiales très régulières sous des hypothèses de type Kalman.

Corollaire 1. *Supposons que l'une des conditions suivantes est vérifiée*

— (ω, T) *vérifie GCC et* $\overline{\omega} \subset \Omega$,
— (ω, T) *vérifie GCC,* $\omega \cap \partial\Omega \neq \emptyset$ *et* $n \leqslant 3$,
alors il existe un contrôle V *in* $C^0([0,T];(L^2(\Omega))^m)$ *tel que la solution* Y *de* (4.7) *correspondant à la condition initiale* $Y(0) = Y^0$ *dans* $(H^{2n-2}_{(0)}(\Omega))^n$ *satisfait* $Y(T,\cdot) = 0$ *si et seulement si* *rang*$(B|AB|A^2B|\cdots|A^{n-1}B) = n$.

Les conditions sur ω et sur le nombre d'équations n permettent de satisfaire les conditions techniques "de contrôlabilité, de régularité et de localité" sur $\mathcal{Q} = \Delta$ dans le Théorème 3. Plus précisement, définissons par $\tilde{\mathbb{1}}_\omega \in \mathcal{C}^\infty(\Omega)$ une fonction telle que $\tilde{\mathbb{1}}_\omega(x) = 0$ pour tout $x \in \Omega\backslash\omega$. Nous devons satisfaire simultanément les conditions suivantes

H1-ondes) l'équation $\partial_{tt} z = \Delta z + \tilde{\mathbb{1}}_\omega u$ est contrôlable à 0 au temps T^* ("contrôlabilité").

H2-ondes) $\Delta^k(\tilde{\mathbb{1}}_\omega \varphi) = 0, \quad$ sur $\quad \partial\Omega, \quad \forall \varphi \in H^{2(k-1)}_{(0)}(\Omega), \quad \forall k \leqslant n-2$ ("régularité et localité"),

avec $H^{2k}_{(0)}(\Omega) := \{v \in H^{2k}(\Omega)$ telle que $v = \Delta v = \cdots = \Delta^{n-1} v = 0$ sur $\partial\Omega\}$. Pour satisfaire la condition **H1-ondes** nous supposons que (ω, T) vérifie GCC qui est une condition (quasi-) nécessaire et suffisante de contrôlabilité pour l'équation des ondes. De plus, la condition **H2-ondes** nous oblige à restreindre le nombre d'équations n lorsque $\omega \cap \partial\Omega \neq \emptyset$. Nous renvoyons à [Chapitre 5, Section 5.4.1] et [29, Section 4.2]) pour plus d'informations.

Appliquons maintenant le Théorème 4 pour un système d'équations de Schrödinger

$$\begin{cases} \partial_t Y = i\Delta Y + AY + \mathbb{1}_\omega BV & \text{dans} \quad (0,T) \times L^2(\Omega)^n, \\ Y(0) = Y^0, \end{cases} \tag{4.8}$$

avec $(A, B) \in \mathcal{M}_n(\mathbb{R}) \times \mathcal{M}_{n,m}(\mathbb{R})$.

Corollaire 2. *Supposons que l'une des conditions suivantes est vérifiée*
— (ω, T) *vérifie GCC et* $\overline{\omega} \subset \Omega$,
— (ω, T) *vérifie GCC,* $\omega \cap \partial\Omega \neq \emptyset$ *et* $n \leqslant 3$,
— *L'ensemble* $\Omega \subset \mathbb{R}^N$ *est le produit de* N *intervalles ouverts et* ω *quelconque,*
— *L'ensemble* $\Omega \subset \mathbb{R}^2$ *est le disque unité,* $\omega \cap \partial\Omega \neq \emptyset$ *et* $n \leqslant 3$.
Alors il existe un contrôle V *in* $C^0([0,T];(L^2(\Omega))^m)$ *tel que la solution* Y *de* (4.8) *correspondant à la condition initiale* $Y(0) = Y^0$ *dans* $(H^{2n-2}_{(0)}(\Omega))^n$ *satisfait* $Y(T,\cdot) = 0$ *si et seulement si* $Y^0 \in (B|AB|A^2B|\cdots|A^{n-1}B)((L^2(\Omega))^{nm})$.

Les conditions sur (ω, T) et sur n sont similaires pour les conditions **H1-ondes** et **H2-ondes**. Grâce à **H1-ondes**, nous ne nous restreignons pas forcément au cas des (ω, T) vérifiant GCC.

4.4 Les méthodes utilisées

Commençons par donner les idées de la preuve du Théorème 4. Nous supposons que $Y^0 \in [A|B](\mathcal{H}^{nm})$ avec

$$[A|B] = (B|AB|A^2B|\cdots|A^{n-1}B).$$

Nous souhaitons montrer que la solution de (Ord1) avec la condition initiale Y^0 peut être amenée à 0 au temps T. Nous allons utiliser une méthode de contrôle fictif introduite dans [38]. L' hypothèse de contrôlabilité scalaire 5.1.1 du Chapitre 5 et le changement de variable $Z = \exp(tA)\tilde{Z}$ nous permet de contrôler la solution du système analytique

$$\begin{cases} \partial_t Z = L_n(Z) + AZ + \overline{C}_n U & \text{in} \quad (0,T) \times \mathcal{H}^n, \\ Z(0) = Y^0. \end{cases} \tag{4.9}$$

avec autant de contrôles que d'équations, c'est à dire $m = n$. Grâce à l'hypothèse $Y^0 \in (\mathcal{D}(\mathcal{L})^{n-1})^n \cap [A|B](\mathcal{H}^{nm})$ et [33] nous construisons un contrôle $U \in [A|B](\mathcal{H}^{nm})$ de (4.9) suffisamment régulier tel que $U(T) = U(0) = 0$, ce qui nous permettra de déterminer une solution (X, W) du système algébrique

$$\partial_t X = L_n(X) + AX + BC_m W + [A|B]\overline{\mathcal{C}}_{mn}\hat{U} \quad \text{dans} \quad (0,T) \times \mathcal{H}^n \tag{4.10}$$

avec

$$X(0) = X(T) = 0, \tag{4.11}$$

et $U = [A|B]\hat{U}$. La résolution du système (4.10) est basée sur une idée de [41, Section 2.3.8] et sera détaillée dans le paragraphe suivant. On conclut en remarquant que le couple $(Y, V) = (Z - X, -W)$ est une solution de (Ord1) qui vérifie bien $Y(0) = Z(0) - X(0) = Y^0$ et $Y(T) = Z(T) - X(T) = 0$. La réciproque est basée sur la preuve de [5, Theorem 1.5]. On renvoie au [Chapitre 5, Section 5.2.2] pour plus de détails.

La démonstration du Théorème 4 est plus complexe à cause de la dérivée d'ordre deux en temps. Par exemple, nous ne pouvons plus appliquer le changement de variable $Z = \exp(tA)\tilde{Z}$ pour le problème analytique, ce qui nous fait perdre la caractérisation des données initiales qui sont contrôlables (voir [Chapitre 5, Proposition 12]).

Résolution du système algébrique (4.10) **:** L'idée provenant de [41, Section 2.3.8] est de réécrire le système (4.10) comme un système sous-déterminé en la variable X et W et de voir $f := \overline{\mathcal{C}}_{mn}\hat{U}$ comme un terme source. Cette idée a été appliquée à la théorie du contrôle dans [25] pour montrer un résultat de contrôlabilité interne pour un système de Navier-Stokes en dimension trois avec un contrôle ayant deux composantes nulles. Cette méthode a ensuite permis de traiter le cas des systèmes paraboliques avec une matrice de couplage d'ordre 0 et d'ordre 1 dans [31] et des systèmes hyperboliques quasi-linéaires dans [2]. Puisque le système (4.10) est composé de n équations avec $n+m$ inconnues, nous avons l'espoir de trouver une solution (X, W) qui satisfait la condition supplémentaire (4.11). Réécrivons le problème (4.10) sous une forme abstraite en introduisant un opérateur différentiel $\mathcal{P}$

$$\mathcal{P}(X, W) = [A|B]\overline{\mathcal{C}}_{mn}\hat{U}, \tag{4.12}$$

où

$$\mathcal{P} : (X\ ,\ W) \mapsto \partial_t X - L_n X - AX - BW. \tag{4.13}$$

Nous construisons un opérateur $\mathcal{M}$ [Chapitre 3, Proposition 11] tel que

$$\mathcal{P} \circ \mathcal{M} = [A|B]. \tag{4.14}$$

L'opérateur $\mathcal{M}$ agit comme une certaine puissance de l'opérateur différentiel $-\partial_t + \mathcal{L}$ [Chapitre 3, (5.29)]. D'après (4.14), $\mathcal{M}(\overline{\mathcal{C}}_{mn}\hat{U}) := (X, W)$ est une solution formelle du système (4.10). La régularité du contrôle $U := [A|B]\hat{U}$ du système (4.9) et l'hypothèse de régularité et de localité 5.1.2 du Chapitre 5 nous permettent de donner un sens à $M(\overline{\mathcal{C}}_{mn}\hat{U}) = (X, W)$ et de montrer que $W \in \mathcal{C}_m\mathcal{U}^m$. La condition $U(T) = U(0) = 0$ implique $X(0) = X(T) = 0$. Nous avons donc construit une solution (X, W) du problème algébrique (4.10) vérifiant (4.11).

Construction de l'opérateur $\mathcal{M}$: En passant à l'adjoint dans (4.14) nous obtenons pour tout

$\varphi \in \mathcal{D}(P^*)$,

$$\mathcal{M}^* \circ P^* = [A|B]^* \quad \text{avec} \quad \mathcal{P}^*\varphi = \begin{pmatrix} -\partial_t\varphi - L_n^*\varphi - A^*\varphi \\ -B^*\varphi \end{pmatrix}. \tag{4.15}$$

Par définition de $[A|B]^*$, pour résoudre (4.15), nous avons besoin de trouver nm relations faisant intervernir $B^*(A^*)^i$ et $\mathcal{P}^*$. Les m premières relations sont données par

$$B^*\varphi = -\begin{pmatrix} (P^*\varphi)_{n+1} \\ \vdots \\ (P^*\varphi)_{n+m} \end{pmatrix}.$$

Nous déterminons les $(n-1)m$ autres relations en multipliant l'équation $-\partial_t\varphi - L_n^*\varphi - A^*\varphi$ (qui correspond au n premières composantes de $\mathcal{P}^*$) par $B^*(A^*)^i$. Nous avons ainsi la relation, pour tout $i \in \{1, \cdots, n-1\}$,

$$B^*(A^*)^i\varphi = -B^*(A^*)^{i-1}\begin{pmatrix} (P^*\varphi)_1 \\ \vdots \\ (P^*\varphi)_n \end{pmatrix} - (\partial_t + L_m^*)(B^*(A^*)^{i-1}\varphi). \tag{4.16}$$

Par récurrence, nous obtenons la relation entre $B^*(A^*)^i$ et $\mathcal{P}^*$ pour tout $i \in \{0, \cdots, n-1\}$, ce qui nous permet de construire l'opérateur $\mathcal{M}^*$ (voir [Chapitre 5, (5.28)] et donc $\mathcal{M}$ (voir [Chapitre 5, (5.29)]). Nous remarquons que l'hypothèse d'indépendance en temps et en espace de la matrice B et A est fondamentale pour que l'égalité (4.16) soit vérifiée.

4.5 Perspectives

Nous allons tout d'abord énoncer quelques extensions possibles de notre travail

— Une extension naturelle serait de rajouter une matrice de diffusion au système (Ord1) et (Ord2). Considérons donc le système

$$\partial_t Y = DL_n(Y) + AY + BC_mV \qquad \text{dans} \quad (0,T) \times \mathcal{H}^n,$$

et

$$\partial_{tt} Y = DQ_n(Y) + AY + BC_mV \qquad \text{in} \quad (0,T) \times \mathcal{H}^n,$$

où D est une matrice constante. L'égalité (4.16) de la Section 4.4 est mise à défaut si la matrice D ne commute pas avec A et B. Par exemple, supposons que D est une matrice diagonalisable à valeurs propres positives. Nous nous attendons à obtenir une condition de Kalman similaire au cas parabolique (voir [8]) du type $\text{rang}[\lambda_p D + A|B] = n$ pour tout $p \in N^*$ où λ_p désigne la p- ième valeur propre de $\mathcal{Q}$.

— Pour des systèmes comme (Ord1) ou (Ord2) nous pouvons nous demander s'il est possible d'amener les l ($l \in [|1, n-1|]$) premières composantes de la solution en 0 sans imposer aucune restriction sur les $n-l+1$ dernières composantes. Il peut être intéressant de déterminer des conditions similaires à [36].

Discutons sur les restrictions imposées par notre méthode sur les données du problème.

— Une question intéressante est la contrôlabilité locale pour des équations semi-linéaires. Malheureusement, pour le cas d'un système d'équations de Schrödinger (4.8), nous avons besoin de diminuer la régularité des données initiales dans le Théorème 2 pour appliquer un théorème de point fixe. Cependant la stratégie utilisée ne permet pas d'obtenir ce résultat puisqu'on a besoin de construire un contrôle régulier pour le problème analytique (4.9). Une solution pour les équations semi-linéaires serait d'utiliser un théorème d'inversion locale de type Nash-Moser comme dans [2].

— Dans le cas d'un système d'équations des ondes (4.7), nous pouvons aussi nous demander s'il est possible de caractériser les données initiales avec notre méthode. Le point difficile se situe dans la résolution du problème analytique : nous devons montrer que si $Y^0 \in [A|B](\mathcal{H}^{nm})$ alors le contrôle U du système

$$\begin{cases} \partial_{tt} Z = \Delta Z + AZ + \tilde{\mathbb{1}}_\omega U & \text{in} \quad (0,T) \times \mathcal{H}^n, \\ Z(0) = Y^0. \end{cases}$$

appartient à $[A|B](\mathcal{H}^{nm})$.

Énonçons maintenant quelques problèmes ouverts

— Pouvons nous généraliser nos résultats de contrôlabilité à des couplages $A(t)$ et $B(t)$ dépendant du temps pour les sytèmes d'équations (Ord1) et (Ord2) ? Nous nous attendons à obtenir la condition de *Silverman-Meadows* (voir [96]) qui est une extension de la condition du rang de Kalman. Dans [8], les auteurs obtiennent cette condition pour des systèmes paraboliques.

— Quelles sont les conditions nécessaires et suffisantes sur la matrice de couplage $A(x)$ dépendant uniquement de l'espace, sur le couple (ω, T) pour que le système (Ord2) soit contrôlable en un temps T ? Nous pouvons aussi nous intéresser à la même question pour le système de Schrödinger (4.6) mais en n'imposant pas forcément que (ω, T) vérifie GCC.

Chapitre 5

A Kalman rank condition for the indirect controllability of coupled systems of linear operator groups

Joint work with **Pierre Lissy**.

Abstract : In this article, we give a necessary and sufficient condition of Kalman type for the indirect controllability of systems of groups of linear operators, under some "regularity and locality" conditions on the control operator that will be made precise later and fit very well the case of distributed controls. Moreover, in the case of first order in time systems, when the Kalman rank condition is not satisfied, we characterize exactly the initial conditions that can be controlled. Some applications to the control of systems of Schrödinger or wave equations are provided. The main tool used here is the *fictitious control method* coupled with the proof of an *algebraic solvability* property for some related underdetermined system and some regularity results.

Keyworlds : controllability of abstract linear semi-groups, indirect controllability of systems, Schrödinger and wave equations, fictitious control method, algebraic solvability.

5.1 Introduction

5.1.1 Presentation of the problem

The problem of controlling coupled systems of partial differential equations has drawn an increasing interest during the last decade, notably in the case of linear parabolic or hyperbolic linear systems, but also of more complex systems that naturally appear in numerous fields, including fluid mechanics, biology, population dynamics, medicine, etc. A very challenging issue for coupled system is the question of *indirect controllability*, which consists in understanding whether it is possible to act on them by a reduced number of controls (i.e. the number of controls is less that the number of equations), and on which equations it is necessary or not to act. This issue is interesting both from a theoretical and practical point of view.

From a theoretical point of view, this question is closely related to a more fundamental question on systems : *which* and *how* information propagates from one equation of the systems to another through the coupling terms, and notably how the coupling terms influence this propagation ? Another related theoretical question comes from the field of inverse problems : using the well-known duality between controllability and observability, the question of controllability is equivalent to understanding if it is possible to recover the initial conditions of all components of an adjoint system thanks to partial observations on the system, giving notably some quantitative informations about the unique continuation properties. To finish, let us mention that, as explained in details in [1, Section 1], indirect controllability is also closely related to insensitizing control and simultaneous control of coupled systems.

Concerning practical applications, indirect controllability is also crucial : for real-life models involving for example different kinds of physical quantities (velocity of a fluid, temperature, etc.), it might be impossible to act directly and all of them, and then it is important to understand if one can act on a complex system by just controlling for example one of the physical variables. Another point is that it is reasonable to try to limit the number of actuators or sensors for cost reasons. Hence, the questions raised here can also be of interest many other fields like automatic and engineering.

Our precise goal in the present work will be to understand how the algebraic structures of the coupling terms and of the control operator influence the properties of indirect controllability for conservative systems. More precisely, our initial motivation was to derive new controllability results in the spirit of [5] (which concerns systems of heat equation) in the case of conservative systems. Let us emphasize that for conservative systems, the strategy developed in [5] (for systems of heat equations) cannot be used at all because it was based on Carleman estimates. In this paper, we propose a possible strategy (that will be described briefly soon) that turns out to be valid in a much more general framework, so that the possible applications of this work cover a large class of problems, basically conservative partial differential equations with internal control.

For the sake of clarity, let us briefly explain on some examples the spirit of the present contribution on two examples, the Schrödinger or wave systems with internal control and constant coupling terms (the detailed results in these cases are given in Section 5.4). Let $T > 0$ and Ω be a smooth bounded open subset of $\mathbb{R}^N$. We denote by $L^2(\Omega)$ the set of square-integrable functions defined on Ω with values in the complex plane $\mathbb{C}$. Let $(n, m) \in (\mathbb{N}^*)^2$ with $n \geqslant 2$. We consider the following control system of n Schrödinger equations with internal control

$$\left\{ \begin{array}{ll} \partial_t Y = i\Delta Y + AY + \mathbb{1}_\omega BV & \text{in} \quad (0,T) \times L^2(\Omega)^n, \\ Y(0) = Y^0 \in L^2(\Omega)^n, & \end{array} \right. \tag{5.1}$$

with $(A, B) \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_{n,m}(\mathbb{C})$, $Y \in L^2(\Omega)^n$ the state and $V \in L^2(\Omega)^m$ the control, that may not act on all the equations.

We also consider the following control system of n wave equations with internal control

$$\left\{ \begin{array}{ll} \partial_{tt} Y = \Delta Y + AY + \mathbb{1}_\omega BV & \text{in} \quad (0,T) \times L^2(\Omega)^n, \\ Y(0) = Y^0 \in H_0^1(\Omega)^n, & \\ \partial_t Y(0) = Y^1 \in L^2(\Omega)^n. & \end{array} \right. \tag{5.2}$$

In this context, a natural issue is the following : is it possible to find necessary and sufficient algebraic conditions on A and B of Kalman type that ensure the null controllability of systems (5.1) or (5.2), under some appropriate geometric conditions on ω (and in sufficiently large time for (5.2)) ? The general method that we will use in the article to answer this question is sometimes called *fictitious control method* and was first introduced in [38]. It has been then used in different context, notably in [2], [25] and [31]. Let us explain the strategy on equation (5.1) (this is the same on equation (5.2)). We first control the equations with n controls (one on each equation) and we try to eliminate $n - m$ controls thanks to algebraic manipulations. More precisely, we decompose the problem into two different steps :

Analytic problem :
Find a solution (Z, V) in an appropriate space to the control problem by n controls which are regular enough, i.e. solve

$$\begin{cases} \partial_t Z = i\Delta Z + AZ + \mathbb{1}_\omega V & \text{in } Q_T, \\ Z = 0 & \text{on } \Sigma_T, \\ Z(0,\cdot) = y^0, \ Z(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{5.3}$$

where $V = (V_1, \dots V_n)$. Solving Problem (5.3) is easier than solving the null controllability at time T of System (5.1), because we control System (5.3) with a control on each equation.
Algebraic problem :
For $f := \mathbb{1}_\omega U$, find a pair (X, W) (where W has now only m components) in an appropriate space satisfying the following control problem :

$$\begin{cases} \partial_t X = i\Delta X + AX + BW + f & \text{in } Q_T, \\ X = 0 & \text{on } \Sigma_T, \\ X(0,\cdot) = X(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{5.4}$$

and such that the spatial support of X is strongly included in ω. We will solve this problem using the notion of *algebraic solvability* of differential systems, which is based on ideas coming from [41, Section 2.3.8]. The idea is to write System (5.4) as an *underdetermined* system in the variables X and W and to see f as a source term, so that we can write Problem (5.4) under the abstract form

$$\mathcal{L}(X, W) = f,$$

where

$$\mathcal{L}(X, W) := \ \partial_t X - i\Delta X - AX - BW.$$

The goal will be then to find a partial differential operator $\mathcal{M}$ satisfying

$$\mathcal{L} \circ \mathcal{M} = Id. \tag{5.5}$$

When (5.5) is satisfied, we say that System (5.4) is *algebraically solvable*. This exactly means that one can find a solution (X, W) to System (5.4) which can be written as a linear combination of some derivatives of the source term f.

Conclusion :
If we can solve the analytic and algebraic problems, then it is easy to check that $(Y, V) = (Z - X, -W)$ will be a solution to System (5.1) in an appropriate space and will satisfy $Y(T) \equiv 0$ in Ω.

For more details concerning this method, we refer to [31, Section 2.3], [25, Section 3.1], [75, Section 1.3] and Sections 5.2 and 5.3. Thanks to this method, we are able, for systems like (5.1) and (5.2) and under some additional conditions, to find some sufficient condition of controllability (the Kalman matrix $[B|AB|\dots A^{n-1}B]$ has to be of maximal rank) that can be proved to be also necessary. In the case of equation 5.1, if the Kalman matrix is not satisfied, the same method also enables us to characterize the initial conditions that can be controlled. The examples of the wave and Schrödinger equations are treated in details in Section 5.4 of this work and all the results presented in in Section 5.4 are new. This approach may be to some extent generalized to abstract linear groups of operators under appropriate assumptions that are explained in details in Section 5.1.2.

Our paper is organized as follows. In Section 5.1.2, we explain in details the abstract framework chosen here and give the main results. In Section 5.1.3, we recall some previous results and explain precisely the scope of the present contribution. In Section 5.1.4 we present some related open problems. Section 5.2 and 5.3 are respectively devoted to proving Theorem 6 and Theorem 7. In Section 5.4, we conclude with some applications, giving new results for the indirect controllability of Schrödinger equations and wave equations with internal control.

5.1.2 Abstract setting and main results

Let us introduce some notations. Let $T > 0$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{U}$, $\mathcal{H}$ two Hilbert spaces on $\mathbb{K}$ (that will be always identified with their dual in what follows) and a linear continuous application $\mathcal{C} : \mathcal{U} \to \mathcal{H}$, which will be our control operator (one may think as an example to consider C as a distributed control). We consider

— $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ a closed unbounded operator with dense domain, which is supposed to be the generator of a strongly continuous **group** on $\mathcal{H}$,

— $\mathcal{Q} : \mathcal{D}(\mathcal{Q}) \subset \mathcal{H} \to \mathcal{H}$ another closed unbounded operator with dense domain, which is supposed to be **self-adjoint and negative with compact resolvent**.

One can think as an example to consider $\mathcal{L} = i\Delta$ and $\mathcal{Q} = \Delta$, which are typical situations where our general result can be applied (see Section 5.4).

Let $n \in \mathbb{N}^*$ with $n \geqslant 2$ and $m \in \mathbb{N}^*$, and let $(A, B) \in \mathcal{M}_n(\mathbb{K}) \times \mathcal{M}_{n,m}(\mathbb{K})$ two matrices. For every $k \in \mathbb{N}^*$, we introduce the operators $L_k : \mathcal{D}(\mathcal{L})^k \subset \mathcal{H}^k \to \mathcal{H}^k$, $Q_k : \mathcal{D}(\mathcal{Q})^k \subset \mathcal{H}^k \to \mathcal{H}^k$ and $C_k : \mathcal{U}^k \to \mathcal{H}^k$ such that, for every $\varphi \in \mathcal{D}(\mathcal{L})^k$ and $\psi \in \mathcal{U}^k$,

$$L_k(\varphi) = \begin{pmatrix} \mathcal{L}(\varphi_1) \\ \mathcal{L}(\varphi_2) \\ \vdots \\ \mathcal{L}(\varphi_k) \end{pmatrix}, \quad Q_k(\varphi) = \begin{pmatrix} \mathcal{Q}(\varphi_1) \\ \mathcal{Q}(\varphi_2) \\ \vdots \\ \mathcal{Q}(\varphi_k) \end{pmatrix} \quad \text{and} \quad C_k(\psi) = \begin{pmatrix} \mathcal{C}(\psi_1) \\ \mathcal{C}(\psi_2) \\ \vdots \\ \mathcal{C}(\psi_k) \end{pmatrix}. \tag{5.6}$$

We consider the (first and second order in time) systems of n linear equations

$$\partial_t Y = L_n(Y) + AY + BC_m V \qquad \text{in} \quad (0, T) \times \mathcal{H}^n, \tag{Ord1}$$

and

$$\partial_{tt} Y = Q_n(Y) + AY + BC_m V \qquad \text{in} \quad (0, T) \times \mathcal{H}^n, \tag{Ord2}$$

where $V := (v_1, \ldots, v_m) \in \mathcal{U}^m$ is called a control. One can think of equation (Ord1) as a generalization of the Schrödinger system (5.1), whereas one can think of equation (Ord2) as a generalization of the wave system (5.2).

(Ord1) (resp. (Ord2)) can be seen as a "system" version of the "scalar" controlled equation $\partial_t z = \mathcal{L}z + \mathcal{C}u$ (resp. $\partial_{tt} z = \mathcal{L}z + \mathcal{C}u$), where we add some coupling terms of zero order through the matrix A and where we impose a precise structure on the control through the matrix B. Note that one may have $m < n$, which means that the number of controls can be strictly less than the number of equations, and notably some equations might be uncontrolled. In this setting, the structure of the coupling terms is crucial in order to obtain some controllability results, and in some sense these coupling terms can be used to act indirectly on the equations that are not controlled.

It is usual to write the second order system (Ord2) as a first order system (see section 5.3.1).

However, we emphasize that (Ord2) *is not* a particular case of (Ord1), the reason being that if we transform (Ord2) into a first order system, we will not be able to find any matrix A such that (Ord2) can be written as (Ord1). Finding such a matrix would require that the coupling terms involve simultaneously Y and Y_t (see notably (5.39)), which is not the case here.

It is well-known that for the controllability of coupled systems like (Ord2), the natural state space $\mathcal{D}(\mathcal{Q}^{\frac{1}{2}})^n \times \mathcal{H}^n$ is not always possible. For instance taking zero as initial data we cannot reach any target state in $\mathcal{D}(\mathcal{Q}^{\frac{1}{2}})^n \times \mathcal{H}^n$ because of the regularity of solutions of (Ord2) (one might for example think of a upper triangular matrix A with a control acting only on the last equation, see [28]). The same phenomena does not occur for system like (Ord1), however, in both cases, we will always assume that the initial conditions are regular enough, namely $Y(0) \in \mathcal{D}(\mathcal{L}^{n-1})^n$ (resp. $(Y(0), \partial_t Y(0)) \in \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \times \mathcal{D}(\mathcal{Q}^{n-1})^n$), which is enough to ensure that system (Ord1) (resp. (Ord2)) with initial condition in these spaces admits a unique solution in $\mathcal{C}^0([0,T];\mathcal{H}^n)$ (resp. in $C^0([0,T];\mathcal{D}(\mathcal{Q}^{\frac{1}{2}})^n \times \mathcal{H}^n)$.

The main goal of this article is to analyze the null controllability of System (Ord1) and System (Ord2), which would (partially) generalize the results of [5] in the case of conservative systems. Let us recall the definition of these notions. It will be said that

- System (Ord1) (resp. System (Ord2)) is *null controllable* at time T if for every initial condition $Y^0 \in \mathcal{D}(\mathcal{L}^{n-1})^n$ (resp. $(Y^0, Y^1) \in \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \times \mathcal{D}(\mathcal{Q}^{n-1})^n$), there exists a control $V \in C^0([0,T];\mathcal{U}^m)$ such that the solution Y to System (Ord1) with initial condition $Y(0) = Y^0$ (resp. to System (Ord2) with initial condition $(Y(0), \partial_t Y(0)) = (Y^0, Y^1)$) satisfies

$$Y(T) \equiv 0 \ \text{ in } \ \mathcal{H}^n \quad (\text{resp. } Y(T) \equiv 0 \textit{ and } \partial_t Y(T) \equiv 0 \ \text{ in } \ \mathcal{H}^n \times \mathcal{H}^n).$$

— System (Ord1) (resp. System (Ord2)) is *exactly controllable* at time T if for every initial condition $Y^0 \in \mathcal{D}(\mathcal{L}^{n-1})^n$ (resp. $(Y^0, Y^1) \in \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \times \mathcal{D}(\mathcal{Q}^{n-1})^n$) and every $Y_T \in \mathcal{D}(\mathcal{L}^{n-1})^n$ (resp. $(Y_T, Z_T) \in \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \times \mathcal{D}(\mathcal{Q}^{n-1})^n$), there exists a control $V \in C^0([0,T];\mathcal{U}^m)$ such that the solution Y to System (Ord1) with initial condition $Y(0) = Y^0$ (resp. to System (Ord2) with initial condition $(Y(0), \partial_t Y(0)) = (Y^0, Y^1)$) satisfies

$$Y(T) \equiv Y_T \ \text{ in } \ \mathcal{D}(\mathcal{L}^{n-1})^n \,(\text{resp. } Y(T) \equiv Y_T \ \text{ in } \ \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \text{ and } \partial_t Y(T) \equiv Z_T \ \text{ in } \ \mathcal{D}(\mathcal{Q}^{n-1})^n).$$

Let us remark that since L_n (*resp.* $\hat{Q}_n$ *defined in* (5.38) , *see Section 5.3.1*) is a generator of a group, then System (Ord1) (resp. System (Ord2)) is null controllable at time T if and only if it is exactly controllable at time T (see for example [24, p. 55]). Hence, from now on, we will only concentrate on the null controllability of Systems (Ord1) and (Ord2).

Our main assumptions (that will be commented afterwards) will be the following.

ASSUMPTIONS

There exists a linear continuous application $\overline{\mathcal{C}} : \mathcal{U} \to \mathcal{H}$ such that

Assumption 5.1.1. [Scalar null controllability]

Case (Ord1) The control system

$$\partial_t z = \mathcal{L} z + \overline{\mathcal{C}} u, \tag{5.7}$$

is exactly controllable at time T^*.

Case (Ord2) The control system

$$\partial_{tt} z = \mathcal{Q} z + \overline{\mathcal{C}} u, \tag{5.8}$$

is exactly controllable at time T^*.

Assumption 5.1.2. [Regularity and locality]
Case (Ord1) $\mathcal{L}^k(\overline{\mathcal{C}}\overline{\mathcal{C}}^*\mathcal{D}(\mathcal{L}^{*k})) \subset \mathcal{C}(\mathcal{U})$ for all $k \in \{0, \cdots, n-1\}$,
Case (Ord2) $\mathcal{Q}^{\frac{k}{2}}(\overline{\mathcal{C}}\overline{\mathcal{C}}^*\mathcal{D}(\mathcal{Q}^{\frac{k}{2}})) \subset \mathcal{C}(\mathcal{U})$ for all $k \in \{0, \cdots, 2n-2\}$.

Remark 1. i It might happen that the operator $\mathcal{C}$ itself verify Assumptions 5.1.1 and 5.1.2, hence in some sense our assumptions are **more general** than just stating the same assumptions replacing $\overline{\mathcal{C}}$ by $\mathcal{C}$. However, the price to pay is that condition 5.1.1 is **stronger** than just having Assumption 5.1.1 with the control operator $\mathcal{C}$ (see the next point).

ii Assumption 5.1.1 may seem quite artificial since it does not seem to be related to the controllability of systems

$$\partial_t z = \mathcal{L}z + \mathcal{C}u, \tag{5.9}$$

and

$$\partial_{tt} z = \mathcal{L}z + \mathcal{C}u, \tag{5.10}$$

that would be the natural minimum conditions one might expect.
However, one can easily prove that Assumptions 5.1.1 and 5.1.2 imply the controllability at time T^* of (5.9) and (5.10). Let us explain it for (5.9) (this exactly the same reasoning for equation (5.10)). Thanks to the Hilbert Uniqueness Method (HUM, see [73]), we know that the control function u in (5.7) with minimal L^2-norm is necessarily in $R(\overline{\mathcal{C}}^*)$ (where R denotes the range of an operator), which implies that $\overline{\mathcal{C}}u \in R(\overline{\mathcal{C}}\overline{\mathcal{C}}^*)$, hence thanks to Assumption 5.1.2 (with $k=0$), we obtain $\overline{\mathcal{C}}u \in R(C)$ and this proves that (5.9) is indeed controllable.

iii One consequence of Assumption 5.1.2 is that we have $\overline{\mathcal{C}}\overline{\mathcal{C}}^*\mathcal{D}(\mathcal{L}^{*k}) \in \mathcal{D}(\mathcal{L}^k)$ for every $k \in \{0, \cdots, n-1\}$, hence is some sense $\overline{\mathcal{C}}$ has to "preserve the regularity", which is very natural in the context of conservative systems of second order like (Ord2) (see notably [29]). However, in many applications this is in general false for the operator $\mathcal{C}$ itself (and it the main reason why we introduce $\overline{\mathcal{C}}$). For example, consider some open subset Ω of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$), and consider $\mathcal{H} = \mathcal{U} = L^2(\Omega)$. Assume that $\mathcal{L}$ is a differential operator defined on some open subset Ω and the application $\mathcal{C} : L^2(\Omega) \to L^2(\Omega)$ is defined by $\mathcal{C}u = \mathbb{1}_\omega u$, where ω is some open subset of Ω. Then it is clear that the property $\mathcal{C}\mathcal{C}^*\mathcal{D}(\mathcal{L}^{*k}) \in \mathcal{D}(\mathcal{L}^k)$, which is equivalent to $\mathbb{1}_\omega u \mathcal{D}(\mathcal{L}^{*k}) \in \mathcal{D}(\mathcal{L}^k)$, is *always false* as soon as $k > 0$, $\mathcal{L}$ is of order more than 1 and ω is different from Ω (because $\mathbb{1}_\omega \in L^2(\Omega)$ but does not belong to any higher order Sobolev space). Hence, roughly speaking, the linear application $\overline{\mathcal{C}}$ has to be thought as a "regularization" of the linear application $\mathcal{C}$. In the case of distributed control, a natural candidate for $\overline{\mathcal{C}}$ is $\overline{\mathcal{C}}u = \tilde{\mathbb{1}}_\omega u$, where $\tilde{\mathbb{1}}_\omega \in \mathcal{C}_c^\infty(\Omega)$ is some "regularization" of the indicator function $\mathbb{1}_\omega$, defined for example such that

$$\tilde{\mathbb{1}}_\omega := \begin{cases} 1 & \text{on} \quad \omega_0, \\ 0 & \text{on} \quad \Omega \backslash \omega, \end{cases}$$

where ω_0 is some well-chosen open set included in ω. This will be explained into more details in Sections 5.4.1 and 5.4.2.

iv Adding the condition $\mathcal{L}^k(\overline{\mathcal{C}}\overline{\mathcal{C}}^*\mathcal{D}(\mathcal{L}^{*k})) \subset \mathcal{C}(\mathcal{U})$ for all $k \in \{0, \cdots, n-1\}$ is necessary in our method to prove that our control is in $(\mathcal{C}(\mathcal{U}))^m$. This notably ensures that the operator $\mathcal{M}$ and $\tilde{\mathcal{M}}$ defined in (5.29) and (5.53) respectively are "local" in the sense that they send an element of the range of $\overline{\mathcal{C}}\overline{\mathcal{C}}^*$ into an element of the range of $\mathcal{C}$.

In the sequel, we will denote by $[A|B] \in \mathcal{M}_{n,nm}(\mathbb{K})$ the Kalman matrix, which is given by

$$[A|B] = (B|AB|A^2B|\cdots|A^{n-1}B). \tag{5.11}$$

Our result gives a necessary and sufficient condition for exact (or null) controllability of System (Ord1) and (Ord2).

Theorem 6. *Let us assume that $\mathcal{L}$ satisfies Assumptions 5.1.1 and 5.1.2 . Let $Y^0 \in (\mathcal{D}(\mathcal{L})^{n-1})^n$. Then, for every $T > T^*$, there exists a control V in $C^0([0,T] \times \mathcal{U}^m)$ such that the solution of (Ord1) corresponding to the initial condition $Y(0) = Y^0$ in $\mathcal{H}^n$ satisfies*

$$Y(T) \equiv 0 \quad in \quad \mathcal{H}^n$$

if and only $Y^0 \in [A|B](\mathcal{H}^{nm})$.

Remark 2. Concerning Theorem 6, the reversibility of the equation allows us to obtain the same conclusion if we replace the final condition $Y(T) = 0$ by $Y(T) = Y^T$ for some $Y^T \in \mathcal{D}(\mathcal{L}^{n-1})^n \cap [A|B](\mathcal{H}^{nm})$.

When $\mathrm{rank}([A|B]) = n$, Theorem 6 gives us a necessary and sufficient condition for the null controllability of System (Ord1).

Corollary 4. *Let us assume that $\mathcal{L}$ satisfies Assumptions 5.1.1 and 5.1.2. Then, for every $T > T^*$, the control system (Ord1) is exactly controllable at time T if and only if rank$([A|B]) = n$.*

Concerning system (Ord2), we have the following result.

Theorem 7. *Let us assume that $\mathcal{Q}$ satisfies assumptions 5.1.1 and 5.1.2. Then, for every $T > T^*$, the control system (Ord2) is exactly controllable at time T if and only if rank$([A|B]) = n$.*

Remark 3. We were not able to derive the same kind of result as in Theorem 6 in the case of second-order in time systems, so that we do not know it would be true in this context.

Using the transmutation technique (for instance the version given in [35]) and Section 5.3.2, one can deduce easily the following result in the parabolic case, assuming that the corresponding "scalar" hyperbolic system is controllable.

Corollary 5. *Let us assume that $\mathcal{Q}$ satisfies Assumptions 5.1.1 and 5.1.2. Then, for every $T > 0$, the control system*

$$\partial_t Y = Q_n(Y) + AY + BC_m V \qquad in \quad (0,T) \times \mathcal{H}^n,$$

is null controllable at time T if and only if rank$([A|B]) = n$.

Remark 4. (a) $\mathrm{rank}([A|B]) = n$ is called the Kalman rank condition by analogy with the finite-dimensional case.

(b) The assumption $T > T^*$ enables us to choose a regular control U in time for the analytic system (5.12) such that $U(0) = U(T) = 0$ (see [33]). This is necessary to ensure that during the resolution of the Algebraic Problem, we can construct a solution X of (5.20) or (5.47) such that $X(0) = X(T) = 0$. However, many of the controllability results known in the literature are either results in arbitrary small time or with an "open" condition on the minimal time of control, hence in practice controlling at any time $T > T^*$ rather that at time exactly T^* will not provide a weaker result than in the scalar case.

5.1.3 State of the art and precise scope of the paper

In all what follows, we will mainly concentrate on systems coupled with zero order terms, on distributed controls and on null or exact controllability results. The case of boundary controls (which are unbounded) is not covered by our abstract setting. However, there is also a huge literature on boundary control, approximate controllability and high order coupling terms for coupled systems (see notably [39], [10], [17] or [31] for some recent contributions).

Concerning second-order parabolic equations, the case of coupled systems of heat equations with same diffusion coefficients and constant or time-dependent coupling terms is well-understood (see notably [5], where an algebraic Kalman rank condition similar to the one of the current article is given). In the case of different diffusion coefficients, a necessary and sufficient condition involving some differential operator related to the Kalman matrix was also given in [8]. This case was treated into more details in [11]. Another result concerning the case of two equations with different diffusion is given in [100], where the author also investigates the case of coupling different dynamics, e.g. a heat and a wave equation. However, as soon as the coupling coefficients depend on the space variable, the situation is far more intricate and in general we only have partial results, essentially with two equations, one control force and in the one-dimensional case (see [6] for example) or in simple geometries like cylinders (see [16]). Let us also mention that the non-linear (and even semi-linear) has not been investigated too much up to now (see for example [7], [38] and [23]). For further informations on this topic, we refer to the recent survey [9].

The case of hyperbolic or dispersive systems seems to have been less studied and the results obtained are somehow quite different from the parabolic ones. Concerning the Schrödinger equation, the recent paper [76] considers the case of a cascade system of 2 equations with one control force under the condition that the coupling region and the control region intersect and verify some technical conditions ensuring that a Carleman estimate can be proved. Concerning systems of wave equation, let us mention [1], where a result of controllability in sufficiently large time for second order in time cascade or bidiagonal systems under coercivity conditions on the coupling terms is given. Another related result is also [4], where the case of two wave equations with one control and a coupling matrix A which is supposed to be symmetric and having some additional technical properties is investigated. Let us also mention a result in the one-dimensional and periodic case proved in [91]. In this last article, the authors also prove a result for the Schrödinger equation in arbitrary dimension on the torus, however they only obtained a result in large time, which is rather counter-intuitive and should be only technical. The case of a cascade system of two wave equations with one control on a compact manifold without boundary was treated in [28], where the author also give a necessary and sufficient condition of controllability depending on the geometry of the control domain and coupling region. Let us emphasize that in the four last references, the results obtained in the case of abstract systems of wave equations can be applied to get some interesting results in the case of abstract heat and Schrödinger equations thanks to the *transmutation method* (see [83], [77] or [35]), leading however to strong (and in general artificial) geometric restrictions on the coupling region and control region. Let us also mention a recent result given in [2], which treats the case of some linear system of two periodic and one-dimensional non-conservative transport equations with same speed of propagation, space-time varying coupling matrix and one control and also a nonlinear case.

Regarding the previous presentation, let us precise the exact scope of this paper, which has a rather different spirit from most of the papers presented before concerning conservative systems.

— Our result is given in a very general setting, since we basically work on some group of operators (which are not necessarily differential operators) with a bounded control operator satisfying some technical conditions that appear to be verified in many cases in practice. Notably, our result fits very well (but is not restricted to) the case of conservative systems of PDEs with distributed control, where no general result was known in the case of constant coupling coefficients.

— Contrary to many results in the literature which concentrates on symmetric matrices, bidiagonal matrices or cascade matrices, our result does not require any structural conditions on the coupling matrix A, nor on the matrix B which is often assumed to be acting only on the last(s) equations. Moreover, we do not have any restriction on the number of equations n we treat. Hence, most of the techniques used in the literature will fail in our case. Another important point is that we are able here to give a necessary and sufficient condition of controllability and we also are able in the one-order in time case to characterize precisely the initial conditions that can be controlled, which -as far as we know- was only known for the finite-dimensional case and for linear second order parabolic systems (see [8]).

— The main restriction is that we work with constant coupling coefficients (this implies that the coupling is made everywhere), which do not cover some interesting cases, notably the case of space-varying coefficients described before. Despite this, we believe that the important degree of generality of the present paper compensates this restriction and that our contribution is of interest in order to have a deeper understanding of the controllability properties of coupled systems.

5.1.4 Some related open problems

Let us address some related open questions and possible extensions of this work.

— In the case of equation (Ord1), Assumption 5.1.2 and the regularity condition on the initial data do not seem to be necessary, and it would be natural to expect the same result by just assuming that Assumption 5.1.1 is true with $\overline{\mathcal{C}} = \mathcal{C}$, but the strategy used here prevented us to get this result. The case of admissible unbounded control operators $\mathcal{B} \in \mathcal{L}_c(U, \mathcal{D}((\mathcal{L}^*)'))$ is also still a widely open question (see [67], where the Kalman rank condition is proved to be necessary in order to obtain the approximate controllability).

— A natural extension of the Kalman rank condition is what is called the *Silverman-Meadows condition* (see [96]) in the case of matrix A and B depending on the time variable, that we did not manage to treat with the same strategy.

— Here the coupling terms are supposed to be constant. It would be interesting to replace them by more general coupling terms. A more challenging issue would be to find an abstract setting that would include conservative partial differential equations with coupling terms depending on the space and time variable, at least in the case where the coupling region intersect the control domain.

— Another interesting question is the case of the local controllability of semi-linear equations, the main difficulty being that due to the difference of regularity between the initial condition and the control, standard inverse mapping theorems or fixed-point theorems cannot be used. A possible remedy would be to use a fixed point strategy of Nash-Moser type as in [2].

— When systems of equations like (Ord1) and (Ord2) are concerned, a very natural question that might appear in many applications is what is called *partial controllability*, which means that we would like (for example) to bring only the first l ($l \in [|1, n-1|]$) components of the state variable to 0 without imposing any conditions the $n-l+1$ last components. It would be interesting to see if general conditions like the one found in [36] can be derived. Another interesting and close problem would be to see if the techniques employed here may be useful for the *synchronisation* are *synchronization by groups* of solutions in the spirit of [69] and [68], by means of bounded controls.

— To finish, one could also investigate more general coupled systems of the form

$$\partial_t Y = DL_n(Y) + AY + BC_m V \qquad \text{in} \quad (0,T)\times\mathcal{H}^n$$

and

$$\partial_{tt} Y = DQ_n(Y) + AY + BC_m V \qquad \text{in} \quad (0,T)\times\mathcal{H}^n,$$

where D is some constant matrix, for example a diagonal matrix with (possibly) distinct coefficients and try to derive results similar to [8].

5.2 Proof of Theorem 6

In the sequel, we focus our attention on the null-controllability of the system (Ord1). Suppose that Assumptions 5.1.1 and 5.1.2 are satisfied and let $T > T^*$. We will always consider some initial condition Y^0 belonging to $\mathcal{D}(\mathcal{L}^{n-1})^n$.

5.2.1 First part of the proof of Theorem 6

In all this section, we assume Y^0 is in $[A|B](\mathcal{H}^{nm})$ and we want to prove that the solution to (Ord1) with initial condition Y^0 can be brought to 0 at time T.

Analytic problem :

We consider the control problem

$$\begin{cases} \partial_t Z = L_n(Z) + AZ + \overline{C}_n(U) & \text{in} \quad (0,T)\times\mathcal{H}^n, \\ Z(0) = Y^0. \end{cases} \tag{5.12}$$

Let us emphasize that in this step we act on *all equations* with n distinct controls, one on each equation. U is here called a *fictitious control* because he will disappear at the end of the reasoning. Let us first prove that (5.12) is controllable and give some regularity results on the control and the solution.

Proposition 10. *If Assumption 5.1.1 is satisfied, then for every $T > T^*$, the control system (5.12) is null controllable at time T. Moreover, one can choose a control U such that*

$$U(t,.) \in [A|B](\mathcal{H}^{nm}) \textit{for every } t \in (0,T), \tag{5.13}$$

$$U \in H_0^{n-1}(0,T;\mathcal{U}^n)\bigcap \cap_{k=0}^{n-1} C^k([0,T];\overline{C}_n^*\mathcal{D}((L_n^*)^{n-1-k}), \tag{5.14}$$

$$Z \in \cap_{k=0}^{n-1} C^k([0,T];\mathcal{D}(L_n^{n-1-k})). \tag{5.15}$$

Proof of Proposition 10. Using the change of variables $Z = e^{tA}\tilde{Z}$ and $U = e^{tA}\tilde{U}$, we obtain that the solution Z of system (5.12) is null controllable at time T if and only if the system

$$\left\{ \partial_t \tilde{Z} = L_n(\tilde{Z}) + \overline{C}_n(\tilde{U}) \quad \text{in} \quad (0,T)\times\mathcal{H}^n, \right. \tag{5.16}$$

is null controllable at time T. Remark that system (5.16) is totally uncoupled. Hence, since the control system (5.7) is controllable at time T, using the definitions of L_n and $\overline{C}_n$ given in (5.6), we easily obtain that the control system (5.16) is controllable at time T.

Moreover $\overline{C}_n \in \mathcal{L}(\mathcal{U}^n,\mathcal{H}^n)$ and the operator L_n is a generator of a group on $\mathcal{H}^n$. Then, applying [33, Corollary 1.5] (with $s = n-1$), we deduce that there exists $(\tilde{Z},\tilde{U})$ a solution of (5.16) such that $\tilde{Z}(T) = 0$ and

$$\tilde{Z} \in \cap_{k=0}^{n-1} C^k([0,T];\mathcal{Z}_{n-1-k}),$$

where $\mathcal{Z}_j$ is defined by induction by

$$\mathcal{Z}_0 = \mathcal{H}^n,\ \mathcal{Z}_j = L_n^{-1}(\mathcal{Z}_{j-1} + \overline{C}_n\overline{C}_n^*(\mathcal{D}((L_n^*)^j))).$$

The spaces $\mathcal{Z}_j$ are in general not known explicitly, however, in our case, using Assumption 5.1.2 (see also the second point of Remark 1), it is clear that notably

$$\overline{\mathcal{C}_n\mathcal{C}_n}^*\mathcal{D}((\mathcal{L}_n^*)^j) \subset \mathcal{D}(\mathcal{L}_n^j),$$

from which we deduce easily by induction that for every $j \in [|1,n|]$, we have

$$\mathcal{Z}_j \subset \mathcal{D}(L_n^j),$$

which establishes (5.15).

Moreover, [33, Theorem 1.4] (with $s=n-1$) notably implies that on can choose $\tilde{U}$ belonging to $H_0^{n-1}(0,T;\mathcal{U}^n)$. Finally, to prove (5.14), it enough to prove that

$$\tilde{U} \in \cap_{k=0}^{n-1} C^k([0,T];\overline{C}_n^*\mathcal{D}((L_n^*)^{n-1-k})),$$

which is an immediate consequence of the proof of Corollary 1.5 of [33, Page 1387] (and notably equality (3.19) in this reference).

It remains us to prove (5.13). As in [33, Equality (1.3)], we fix $\delta > 0$ such that $T-2\delta > T^*$ and we consider $\eta \in C^{n-1}(\mathbb{R})$ such that

$$\eta(t) = \begin{cases} 0 & \text{if } t \in (0,T), \\ 1 & \text{if } t \in [\delta, T-\delta]. \end{cases}$$

Then, $\tilde{U}$ can also be chosen as the one of minimal $L^2(0,T;dt/\eta;U)$ among all possible controls for which the solution of (5.12) satisfies $\tilde{Z}(T)=0$, properties (5.14) and (5.15) being still verified.

Hence, using [33, Proposition 1.3], $\tilde{U}$ can be written as

$$\tilde{U} = \eta(t)\overline{C}_n^* e^{(T-t)L_n^*}G_n^{-1}e^{TL_n}Y^0,$$

where

$$G_n = \int_0^T e^{(T-t)L_n}\overline{C_nC_n}^* e^{(T-t)L_n^*}\,dt.$$

Note that since (5.16) is null controllable, G_n is indeed an invertible linear application. Using that $Y^0 = [A|B]\hat{Y}^0$ with $\hat{Y}^0 \in \mathcal{H}^{nm}$, and the formulas

$$\begin{aligned} L_n[A|B] &= [A|B]L_{nm}, \\ \overline{C}_n^*[A|B] &= [A|B]\overline{C}_{nm}^*, \\ \overline{C}_n[A|B] &= [A|B]\overline{C}_{nm}, \end{aligned}$$

we deduce

$$\tilde{U} = [A|B]\eta(t)\overline{C}_{nm}^* e^{(T-t)L_{nm}^*}G_{nm}^{-1}e^{TL_{nm}}\hat{Y}^0,$$

where

$$G_{nm} = \int_0^T e^{(T-t)L_{nm}}\overline{C_{nm}C_{nm}}^* e^{(T-t)L_{nm}^*}\,dt,$$

which is also invertible. Using that $\tilde{U} = e^{-tA}U$, we obtain

$$U = e^{tA}[A|B]\eta(t)\overline{C}_{nm}^* e^{(T-t)L_{nm}^*}G_{nm}^{-1}e^{TL_{nm}}\hat{Y}^0. \tag{5.17}$$

By the Cayley-Hamilton theorem, there exists $\beta = \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{n-1} \end{pmatrix} \in \mathbb{K}^n$ such that $A^n = \sum_{i=0}^{n-1} \beta_i A^i$.

Let $\psi = \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_{n-1} \end{pmatrix} \in \mathcal{H}^{nm}$, we have

$$\begin{array}{rcl} A[A|B]\psi & = & (AB, A^2B, \cdots, A^nB)\psi \\ & = & (AB, A^2B, \cdots, \sum_{i=0}^{n-1} \beta_i A^i B)\psi \\ & = & [A, B]\hat{\psi} \end{array} \tag{5.18}$$

with

$$\hat{\psi} = \begin{pmatrix} \beta_0\psi_{n-1} \\ \psi_0 + \beta_1\psi_{n-1} \\ \vdots \\ \psi_{n-2} + \beta_{n-1}\psi_{n-1} \end{pmatrix} \in \mathcal{H}^{nm}.$$

Combining (5.17), (5.18) and using the fact that there exists $\alpha \in \mathbb{K}^n$ such that $e^{-tA} = \sum_{i=0}^{n-1} \alpha_i A^i$, we obtain (5.13) and the proof of Proposition 10 is complete.

■

Algebraic problem :

Now, we would like to come back to the original system (Ord1) by algebraic manipulations.

Using Proposition 10, there exists (Z, U) solution of (5.12) verifying moreover (5.13), (5.14) and (5.15). Notably, there exists $\hat{U} \in \mathcal{U}^{nm}$ such that $\overline{C}_n(U) = [A|B]\overline{C}_{nm}(\hat{U})$. From now on, we will call $f := \overline{C}_{nm}(\hat{U})$, that will be considered as a source term. Our goal will be to find a pair $(X, \tilde{W}) \in C^0([0, T]; \mathcal{D}(L_n)) \cap C^1([0, T]; \mathcal{H}^n) \times C^0([0, T]; \mathcal{U}^m)$ satisfying the following problem :

$$\left\{ \begin{array}{ll} \partial_t X = L_n(X) + AX + BC_m\tilde{W} + [A|B]f & \text{in} \quad (0, T) \times \mathcal{H}^n, \\ X(0) = X(T) = 0. \end{array} \right. \tag{5.19}$$

Calling $C_m\tilde{W} = W$, we will rather solve (the unknowns being the variables X and W) the following problem

$$\left\{ \begin{array}{ll} \partial_t X = L_n(X) + AX + BW + [A|B]f & \text{in} \quad (0, T) \times \mathcal{H}^n, \\ X(0) = X(T) = 0, \end{array} \right. \tag{5.20}$$

(the fact that $W \in C_m\mathcal{U}^m$ will be a consequence of our construction and our assumptions). Let us remark that system (5.20) is *underdetermined* in the sense that we have more unknowns than equations. Hence, one can hope to find a trajectory (X, W) verifying $X(0) = X(T) = 0$ (which is a *crucial point here*), which would not be necessarily possible if the system were well-posed. We will use the notion of *algebraic solvability*, which is based on ideas coming from [41, Section 2.3.8] for differential systems and was already widely used in [2], [25] and [31]. The idea is to write System (5.20) as an undertermined system in the variables X and W and to see f as a source term, so that we can write Problem (5.20) under the abstract form

$$\mathcal{P}(X, W) = [A|B]f, \tag{5.21}$$

where

$$\mathcal{P}: \begin{array}{lcc} \mathcal{D}(\mathcal{P}) \subset L^2(0,T;\mathcal{H}^{n+m}) & \to & L^2(0,T;\mathcal{H}^n) \\ (X\,,\,W) & \mapsto & \partial_t X - L_n X - AX - BW. \end{array} \tag{5.22}$$

The goal will be then to find an operator $\mathcal{M}$ (involving time derivatives and powers of L_n) satisfying

$$\mathcal{P} \circ \mathcal{M} = [A|B]. \tag{5.23}$$

When (5.23) is satisfied, we say that System (5.20) is *algebraically solvable*. In this cas, one can choose as a particular solution of (5.20) $(X,W) = \mathcal{M}(\overline{C}_{nm}(\hat{U}))$. This exactly means that one can find a solution (X,W) of System (5.20) which can be written as a linear combination f, its derivatives in time, and some $L^k_{nm} f$ with $k \in \mathbb{N}^*$. Let us prove the following Proposition :

Proposition 11. *Let $(A,B) \in \mathcal{M}_n(\mathbb{K}) \times \mathcal{M}_{n,m}(\mathbb{K})$. There exists an operator $\mathcal{M}$ such that the equality (5.23) is satisfied. Moreover, the operator $\mathcal{M}$ is an operator of order at most*

$$\begin{cases} n-2 & \text{for the } n \text{ first components} \\ n-1 & \text{for the } m \text{ last components} \end{cases} \tag{5.24}$$

in time and in term of powers of L_n.

Proof of Proposition 11. We can remark that equality (5.23) is equivalent to

$$\mathcal{M}^* \circ P^* = [A|B]^*. \tag{5.25}$$

The adjoint operator $\mathcal{P}^* : \mathcal{D}(P^*) \subset L^2(0,T;\mathcal{H}^n) \to L^2(0,T;\mathcal{H}^n) \times L^2(0,T;\mathcal{H}^m)$ of the operator $\mathcal{P}$ is given for all $\varphi \in \mathcal{D}(P^*)$ by

$$\mathcal{P}^*\varphi \quad := \begin{pmatrix} (P^*\varphi)_1 \\ \vdots \\ (\mathcal{P}^*\varphi)_n \\ (\mathcal{P}^*\varphi)_{n+1} \\ \vdots \\ (P^*\varphi)_{n+m} \end{pmatrix} = \begin{pmatrix} -\partial_t\varphi - L_n^*\varphi - A^*\varphi \\ -B^*\varphi \end{pmatrix}. \tag{5.26}$$

Since $(A,B) \in \mathcal{M}_n(\mathbb{K}) \times \mathcal{M}_{n,m}(\mathbb{K})$ are constant matrices, we have the following commutation properties :

$$B^*(A^*)^i L_n^* = L_m^* B^*(A^*)^i$$

and

$$B^*(A^*)^i \partial_t = \partial_t B^*(A^*)^i.$$

By definition, we have

$$B^*\varphi = -\begin{pmatrix} (P^*\varphi)_{n+1} \\ \vdots \\ (P^*\varphi)_{n+m} \end{pmatrix}.$$

Now, for $i=\{1,\cdots,n-1\}$, applying $B^*(A^*)^{i-1}$ to $-\partial_t\varphi - L_n^*\varphi - A^*\varphi$, we have

$$B^*(A^*)^{i-1}\begin{pmatrix}(P^*\varphi)_1\\ \vdots\\ (P^*\varphi)_n\end{pmatrix} = -(\partial_t+L_m^*)(B^*(A^*)^{i-1}\varphi) - B^*(A^*)^i\varphi, \text{ i.e.}$$

$$B^*(A^*)^i\varphi = -B^*(A^*)^{i-1}\begin{pmatrix}(P^*\varphi)_1\\ \vdots\\ (P^*\varphi)_n\end{pmatrix} - (\partial_t+L_m^*)(B^*(A^*)^{i-1}\varphi).$$

By induction, we find, for every $i\in\{1,\cdots,n-1\}$,

$$\begin{aligned} B^*(A^*)^i\varphi &= \textstyle\sum_{j=0}^{i-1}(-1)^{j+1}\left((\partial_t+L_m^*)^j B^*(A^*)^{i-1-j}\begin{pmatrix}(P^*\varphi)_1\\ \vdots\\ (P^*\varphi)_n\end{pmatrix}\right) \\ &\quad +(-1)^{i+1}(\partial_t+L_m^*)^i\begin{pmatrix}(P^*\varphi)_{n+1}\\ \vdots\\ (P^*\varphi)_{n+m}\end{pmatrix}. \end{aligned} \tag{5.27}$$

We introduce the operator $\mathcal{M}^*:\mathcal{D}(\mathcal{M}^*)\subset L^2(0,T;\mathcal{H}^{n+m})\to L^2(0,T;\mathcal{H}^{nm})$ defined by

$$\mathcal{M}^*\begin{pmatrix}\psi_1\\ \vdots\\ \psi_{n+m}\end{pmatrix} := \begin{pmatrix} -\psi_{n+1}\\ \vdots\\ -\psi_{n+m}\\ -B^*\begin{pmatrix}\psi_1\\ \vdots\\ \psi_n\end{pmatrix}+(\partial_t+L_m^*)\begin{pmatrix}\psi_{n+1}\\ \vdots\\ \psi_{n+m}\end{pmatrix}\\ \vdots\\ \sum_{j=0}^{n-2}(-1)^{j+1}\left((\partial_t+L_m^*)^jB^*(A^*)^{n-2-j}\begin{pmatrix}\psi_1\\ \vdots\\ \psi_n\end{pmatrix}\right)+(-1)^n(\partial_t+L_m^*)^{n-1}\begin{pmatrix}\psi_{n+1}\\ \vdots\\ \psi_{n+m}\end{pmatrix}\end{pmatrix}. \tag{5.28}$$

Thanks to (5.27) and (5.28), $\mathcal{M}^*$ verifies equality (5.25). Using the definition of $\mathcal{M}^*$ given in

(5.28), we deduce that

$$\begin{array}{rcl}\mathcal{M}: \quad \mathcal{D}(\mathcal{M}) \subset L^2(0,T;\mathcal{H}^{nm}) & \to & L^2(0,T;\mathcal{H}^{n+m}) \\ f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \\ f_{m+1} \\ \vdots \\ f_{2m} \\ \vdots \\ f_{nm} \end{pmatrix} & \mapsto & \mathcal{M}f, \end{array}$$

defined by

$$\mathcal{M}f = \begin{pmatrix} \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} (-1)^{j+1} (-\partial_t + L_n)^j A^{i-1-j} B \begin{pmatrix} f_{jm+1} \\ \vdots \\ f_{(j+1)m} \end{pmatrix} \\ \sum_{i=0}^{n-1} (-1)^{i+1} (-\partial_t + L_m)^i \begin{pmatrix} f_{im+1} \\ \vdots \\ f_{(i+1)m} \end{pmatrix} \end{pmatrix}, \tag{5.29}$$

satisfies (5.23). Thus, in the n first components the higher order term is $(-\partial_t + \mathcal{L})^{n-2}$ and in the m last components the higher order term is $(-\partial_t + \mathcal{L})^{n-1}$, which concludes the proof. ∎

Conclusion : combination of the Analytic and Algebraic Problems.

Thanks to Proposition 10, there exists (Z,U) solution of (5.12) verifying moreover (5.13), (5.14) and (5.15). One can notably write

$$\overline{C}_n(U) = [A|B]\overline{C}_{nm}(\hat{U}) \tag{5.30}$$

for some $\hat{U} \in L^2((0,T);\mathcal{U}^{nm})$. Using Proposition 11, we define (X,W) by

$$\begin{pmatrix} X \\ W \end{pmatrix} := \mathcal{M}\left(\overline{C}_{nm}\hat{U}\right), \tag{5.31}$$

where $\hat{U}$ is defined in (5.30). Using (5.14), we know that

$$\overline{C}_n U \in H_0^{n-1}(0,T;\overline{C}_n(\mathcal{U}^n)) \bigcap \cap_{k=0}^{n-1} C^k([0,T];\overline{C}_n\overline{C}_n^*\mathcal{D}((L_n^*)^{n-1-k})),$$

which implies using Assumption 5.1.2 that

$$\overline{C}_n U \in H_0^{n-1}(0,T;\overline{C}_n(\mathcal{U}^n)) \bigcap \cap_{k=0}^{n-1} C^k([0,T];(\mathcal{D}(\mathcal{L}^{n-1-k}))^n).$$

Using now (5.30), we obtain

$$\overline{C}_{nm}\hat{U} \in H_0^{n-1}(0,T;\overline{C}_{nm}(\mathcal{U}^{nm})) \bigcap \cap_{k=0}^{n-1} C^k([0,T];(\mathcal{D}(\mathcal{L}^{n-1-k}))^{nm}). \tag{5.32}$$

Using (5.31) together with (5.24) and (5.2.1), we obtain that

$$(X,W)\in\left(H_0^1(0,T;\overline{C}_n(\mathcal{U}^n))\bigcap\cap_{k=0}^1 C^k([0,T];\mathcal{D}(L_n^{1-k}))\right)\times\left(L^2(0,T;\overline{C}_m(\mathcal{U}^m))\bigcap C^0([0,T];\mathcal{H}^m)\right),$$

i.e.

$$(X,W)\in\left(H_0^1(0,T;\overline{C}_n(\mathcal{U}^n))\bigcap\cap_{k=0}^1 C^k([0,T];\mathcal{D}(L_n^{1-k}))\right)\times C^0([0,T];\overline{C}_m(\mathcal{U}^m)). \tag{5.33}$$

Notably, there exists $\tilde{W}\in C^0([0,T];\mathcal{U}^m)$ such that $W=C_m\tilde{W}$. Thus, coming back to (5.19), we infer that $(X,\tilde{W})$ is a solution to the problem

$$\begin{cases}\partial_t X=L_n(X)+AX+BC_m\tilde{W}+[A|B]f & \text{in}\quad(0,T)\times\mathcal{H}^n,\\ X(0)=X(T)=0.\end{cases} \tag{5.34}$$

Hence, combining (5.12), (5.34) and the regularity results given in (5.15) and (5.33), the fictitious control f disappears and the pair $(Y,V):=(Z-X,-\tilde{W})$ is a solution to System (Ord1) in the space

$$C^0([0,T];\mathcal{D}(L_n))\cap C^1([0,T];\mathcal{H}^n)\times C^0([0,T];\mathcal{U}^m)$$

satisfying

$$\begin{aligned}Y(0)=Y^0\ \text{ in }\ \mathcal{D}(\mathcal{L}^{n-1})^n,\\ Y(T)\equiv 0\ \text{ in }\ \mathcal{H}^n,\end{aligned}$$

which concludes the first part of the proof of Theorem 6.

■

5.2.2 Second part of the proof of Theorem 6

In this section, we assume Y^0 is NOT in $[A|B](\mathcal{H}^{nm})$ and we want to prove that we cannot bring the solution of (Ord1) with initial Y^0 to 0. We argue by contradiction.

For the sake of completeness we mimic the proof of [5, Theorem 1.5]. Without loss of generality we can only consider the case where we have one control force $m=1$, that is to say $B\in\mathcal{M}_{n,1}(\mathbb{K})$ $(m=1)$, the general case being quite similar (see notably [5, Lemma 3.1 and Page 14]).

Let $l\in\mathbb{N}$ such that $\operatorname{rank}[A|B]=l<n$. By the Cayley-Hamilton theorem, $\{B,AB,\cdots,A^{l-1}B\}$ is linearly independent. We introduce

$$X=\operatorname{span}\{B,AB,\cdots,AB^{l-1}\}.$$

Since $A^lB\in X$, we know that there exists $\alpha\in\mathbb{K}^l$ such that

$$A^lB=\alpha_1B+\alpha_2AB+\cdots+\alpha_lA^{l-1}B. \tag{5.35}$$

Let $p_{l+1},\cdots,p_n$ be $n-l$ vectors in $\mathbb{K}^n$ such that the set

$$\{B,AB,\cdots,A^{l-1}B,p_{l+1},\cdots,p_n\}$$

is a basis of $\mathbb{K}^n$. Introducing

$$P=(B|AB|\cdots|A^{l-1}B|p_{l+1}|\cdots,p_n),$$

we have $Pe_1 = B$ with $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and

$$P^{-1}AP = \begin{pmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{pmatrix}$$

for some $D_{12} \in \mathcal{M}_{l,n-l}(\mathbb{K}), D_{22} \in \mathcal{M}_{n-l}(\mathbb{K})$ and $D_{11} \in \mathcal{M}_l$ which is given by

$$D_{11} = \begin{pmatrix} 0 & 0 & 0 & \cdots & \alpha_1 \\ 1 & 0 & 0 & \cdots & \alpha_2 \\ 0 & 1 & 0 & \cdots & \alpha_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_l \end{pmatrix}$$

By the change of variables $W = P^{-1}Y$ and using $L_n P^{-1} = P^{-1}L_n$ and $\partial_t P^{-1} = P^{-1}\partial_t$ we observe that there exists a control V in $L^2((0,T) \times \mathcal{H})$ such that the solution of (Ord1) corresponding to the initial condition $Y(0) = Y^0$ in $\mathcal{H}^n$ satisfies $Y(T) \equiv 0$ in $\mathcal{H}^n$ if and only if the solution $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$ to

$$\begin{cases} \partial_t W = L_n W + \begin{pmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{pmatrix} W + C_1 V e_1, \\ W(0) = W^0 := P^{-1}Y^0 \end{cases} \tag{5.36}$$

verifies $W(T) \equiv 0$ in $\mathcal{H}^n$. Besides, it is easy to see that $Y^0 \in [A|B](\mathcal{H}^{nm})$ if and only if there exists $W_1^0 \in \mathcal{H}^l$ such that $Y^0 = P\begin{pmatrix} W_1^0 \\ 0 \end{pmatrix}$. If $Y^0 \notin [A|B](\mathcal{H}^{nm})$ then $Y^0 = P\begin{pmatrix} W_1^0 \\ W_2^0 \end{pmatrix}$ with $W_1^0 \in \mathcal{H}^l$, $W_2^0 \in \mathcal{H}^{n-l}$ and $W_2^0 \neq 0$. Thus, by uniqueness we conclude that $W_2(T) \neq 0$. Hence the solution W of (5.36) cannot be driven to zero at time T and (Ord1) cannot driven from Y^0 at time T to 0 at time T, which concludes the proof.

■

5.3 Proof of Theorem 7

Let us recall that we consider here an operator $\mathcal{Q}$ which is assumed to be self-adjoint, negative with compact resolvent. Suppose that Assumptions 5.1.1 and 5.1.2 are satisfied and let $T > T^*$. We will always consider some initial condition $(Y^0, Y^1) \in \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \times \mathcal{D}(\mathcal{Q}^{n-1})^n$.

During this section, we will assume that $\mathbb{K} = \mathbb{C}$ for the sake of simplicity. The case $\mathbb{K} = \mathbb{R}$ can then be easily deduced for example by complexifying the spaces $\mathcal{H}$ and $\mathcal{U}$.

5.3.1 First part of the proof of Theorem 7

In this section, we assume that the Kalman rank condition is satisfied and we want to prove the controllability of (Ord2). We proceed as in the proof of Theorem 6. Let us emphasize that the main difference with the previous case is that the changing of variables exhibited during the proof of Proposition 10 does not work anymore, hence we have to change totally the proof of the analytic part, which will now rely on a classical compactness-uniqueness argument similar to the one given for example in [21, Section 3]. Concerning the algebraic part, the computations are essentially the same.

Analytic problem :

We consider the controlled system

$$\left\{\begin{array}{ll} \partial_{tt}Z = Q_n(Z) + AZ + \overline{C}_n(U) & \text{in} \quad (0,T)\times\mathcal{H}^n, \\ Z(0) = Z^0, & \\ \partial_t Z(0) = Z^1. & \end{array}\right. \tag{5.37}$$

Let us first introduce some notations and first-order framework. Let H_α be the Hilbert space defined by $\mathcal{H}_\alpha = \mathcal{D}(\mathcal{Q}^\alpha)$ for any $\alpha \geqslant 0$ and $H_{-\alpha}$ is the dual space of H_α with respect to the pivot space $\mathcal{H}$. We denote by $X = \left(\mathcal{H}_{\frac{1}{2}}\times\mathcal{H}\right)^n$ our state space. We introduce the operator $\hat{Q} : \mathcal{D}(\hat{Q}) = \left(\mathcal{H}_1\times\mathcal{H}_{\frac{1}{2}}\right)^n \subset X \to X$ such that

$$\hat{Q} = \begin{pmatrix} A_{\mathcal{Q}} & 0 & \cdots & 0 \\ 0 & A_{\mathcal{Q}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_{\mathcal{Q}} \end{pmatrix} \quad \text{with} \quad A_{\mathcal{Q}} = \begin{pmatrix} 0 & I_d \\ \mathcal{Q} & 0 \end{pmatrix}. \tag{5.38}$$

The system (5.37) can be written as a first order system

$$\hat{Z}_t = (\hat{Q}+\hat{A})\hat{Z} + \hat{C}U, \tag{5.39}$$

with $\hat{Z} = \begin{pmatrix} Z_1 \\ (Z_1)_t \\ \vdots \\ Z_n \\ (Z_n)_t \end{pmatrix}$, $\hat{C}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{C}u_1 \\ \vdots \\ 0 \\ \overline{C}u_n \end{pmatrix}$ and $\hat{A} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{11} & 0 & a_{12} & \cdots & 0 & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{n1} & 0 & a_{n2} & \cdots & 0 & a_{nn} \end{pmatrix}$.

Since we identify $\mathcal{H}$ with its dual, we shall define $\left(\mathcal{H}_{\frac{1}{2}}\times\mathcal{H}\right)' = \mathcal{H}\times\mathcal{H}_{-\frac{1}{2}}$ and the duality product defined for $(y_0,y_1)\in\mathcal{H}\times\mathcal{H}_{-\frac{1}{2}}$, $(z_0,z_1)\in\mathcal{H}_{\frac{1}{2}}\times\mathcal{H}$ by

$$\left\langle \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\rangle_{\left(\mathcal{H}\times\mathcal{H}_{-\frac{1}{2}}\right)\times\left(\mathcal{H}_{\frac{1}{2}}\times\mathcal{H}\right)} = \langle y_0, z_1\rangle_H + \langle y_1, z_0\rangle_{\mathcal{H}_{-\frac{1}{2}}\times\mathcal{H}_{\frac{1}{2}}}.$$

With this scalar product, we have

$$\hat{Q}^* = -\begin{pmatrix} A_{\mathcal{Q}^*} & 0 & \cdots & 0 \\ 0 & A_{\mathcal{Q}^*} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_{\mathcal{Q}^*} \end{pmatrix} \quad \text{with} \quad \left\{\begin{array}{l} X' = (\mathcal{H}\times\mathcal{H}_{-\frac{1}{2}})^n, \\ \mathcal{D}(\hat{Q}^*) = \left(\mathcal{H}_{\frac{1}{2}}\times\mathcal{H}\right)^n, \end{array}\right.$$

and $\hat{C}^* : X^* \to \mathcal{U}^n$ is given by

$$\hat{C}^* \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n} \end{pmatrix} = \begin{pmatrix} \overline{\mathcal{C}^*} x_1 \\ \overline{\mathcal{C}}^* x_3 \\ \vdots \\ \overline{\mathcal{C}}^* x_{2n-1} \end{pmatrix}. \tag{5.40}$$

Thus, $\hat{C}\hat{C}^* : X^* \to X$ is exactly

$$\hat{C}\hat{C}^* = \begin{pmatrix} B_{\overline{\mathcal{C}}} & 0 & \cdots & 0 \\ 0 & B_{\overline{\mathcal{C}}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & B_{\overline{\mathcal{C}}} \end{pmatrix} \quad \text{with} \quad B_{\overline{\mathcal{C}}} = \begin{pmatrix} 0 & 0 \\ \overline{\mathcal{C}\mathcal{C}^*} & 0 \end{pmatrix}.$$

and for $i \in \mathbb{N}^*$, $\mathcal{D}(\hat{Q}^i) = (\mathcal{H}_{\frac{i+1}{2}} \times \mathcal{H}_{\frac{i}{2}})^n$ and $\mathcal{D}((\hat{Q}^*)^i) = \mathcal{D}(\hat{Q}^{i-1})$.
We can now go back to the resolution of the analytic problem.
For any operator $\mathcal{R}$, we introduce the set

$$\begin{aligned} \mathcal{N}_{\mathcal{R}}(T) &:= \left\{ \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \in \mathcal{H}_{\frac{1}{2}} \times \mathcal{H} \,|\, C^* z_t = 0 \ \ \forall t \in [0,T], \ \ z \, \text{solution of} \left\{ \begin{array}{l} \partial_{tt} z = \mathcal{R} z \\ z(0) = z^0, \, \partial_t z(0) = z^1 \end{array} \right. \right\}, \\ &= \left\{ \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \in \mathcal{H}_{\frac{1}{2}} \times \mathcal{H} \,|\, (0, C^*) Z = 0 \ \ \forall t \in [0,T], \ \ Z \, \text{solution of} \left\{ \begin{array}{l} \partial_t Z = \begin{pmatrix} 0 & I_d \\ \mathcal{R} & 0 \end{pmatrix} Z \\ Z(0) = \begin{pmatrix} z^0 \\ z^1 \end{pmatrix} \end{array} \right. \right\}. \end{aligned} \tag{5.41}$$

The main proposition is the following.

Proposition 12. *. If Assumption 5.1.1 is satisfied, for every $T > T^*$, the control system (5.37) is null controllable at time T. Moreover, one can choose U such that*

$$U \in H_0^{2n-2}(0,T;\mathcal{U}^n) \bigcap \cap_{k=0}^{2n-2} C^k([0,T], \overline{\mathcal{C}}^* (\mathcal{H}_{n-1-\frac{k}{2}})^n). \tag{5.42}$$

Proof of Proposition 12.
We will need the two following lemmas.

Lemma 20. *If $\mathcal{N}_{Q_n^* + A^*}(T) = \{0\}$ and if the system*

$$\partial_{tt} Z = Q_n(Z) + \overline{C}_n(U) \qquad in \quad (0,T) \times \mathcal{H}^n \tag{5.43}$$

is exactly controllable at time T then the system (5.37) is exactly controllable in time T.

Lemma 20 can be found in [21, Theorem 4] (with $\epsilon = 1$), its proof will then be omitted.

Lemma 21. *If $\mathcal{N}_{Q^*}(T) = \{0\}$ then $\mathcal{N}_{Q_n^* + A^*}(T) = \{0\}$.*

Let us temporarily admit this lemma and explain how we can deduce Proposition 12. By Assumption 5.1.1, we have a unique continuation property for the adjoint system of (5.43), from which we obtain that $\mathcal{N}_{Q^*}(T) = \{0\}$. Now, using Lemma 20 and Lemma 21, the system (5.37) is null controllable at time T. Since $\hat{\mathcal{C}} \in \mathcal{L}(\mathcal{U}^n, X)$, the control system (5.39) is null controllable at time T and the operator $\hat{Q} + \hat{A}$ is a generator of strongly continuous group on

X. Let $T > T^*$, combining one more time Theorem 1.4, Corollary 1.5 and the equality (3.19) in [33] (with $s = 2n-2$), if $Y_0 \in \mathcal{D}(\hat{Q}^{2n-2})$ one can choose U such that

$$U \in H_0^{2n-2}(0,T;\mathcal{U}^n) \bigcap \cap_{k=0}^{2n-2} C^k([0,T]; \hat{C}^*\mathcal{D}((\hat{Q}^*)^{2n-2-k})).$$

Since for $i \in \mathbb{N}^*$, $\mathcal{D}(\hat{Q}^i) = (\mathcal{H}_{\frac{i+1}{2}} \times \mathcal{H}_{\frac{i}{2}})^n$, $\mathcal{D}((\hat{Q}^*)^i) = \mathcal{D}(\hat{Q}^{i-1})$, and going back to the definition of $\hat{C}^*$ given in (5.40), we obtain (5.42).

■

It remains us to prove Lemma 21. The proof is based on the following property.

Lemma 22. *Let $a \in \mathbb{K}$. If $\mathcal{N}_{\mathcal{Q}^*}(T) = \{0\}$ then $\mathcal{N}_{\mathcal{Q}^*+aI_{\mathcal{H}}}(T) = \{0\}$.*

Proof of Lemma 22. Let us decompose the proof into three steps.

- If $C^*\varphi \neq 0$ for every eigenvector φ of $\mathcal{Q}^* + aI_{\mathcal{H}}$ then $\mathcal{N}_{\mathcal{Q}^*+aI_{\mathcal{H}}} = \{0\}$.

  To prove this property we refer to the proof of [21, Theorem 5] which relies on an easy compactness-uniqueness argument.
- If φ an eigenvector of $\mathcal{Q}^*$ associated to the eigenvalue $-\lambda < 0$ and $\mathcal{N}_{\mathcal{Q}^*}(T) = 0$ then $C^*\varphi \neq 0$.

  Indeed, in this case $\begin{pmatrix} \frac{\varphi}{i\sqrt{\lambda}} \\ \varphi \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 0 & I_d \\ \mathcal{Q}^* & 0 \end{pmatrix}$ associated to the eigenvalue $i\sqrt{\lambda}$. Let us assume that $(0, C^*)\begin{pmatrix} \frac{\varphi}{i\sqrt{\lambda}} \\ \varphi \end{pmatrix} = C^*\varphi = 0$. Let Z_φ be the solution of

  $$\begin{cases} \partial_t Z_\varphi = \begin{pmatrix} 0 & I_d \\ \mathcal{Q}^* & 0 \end{pmatrix} Z_\varphi, \\ Z_\varphi(0) = \begin{pmatrix} \frac{\varphi}{i\sqrt{\lambda}} \\ \varphi \end{pmatrix}. \end{cases}$$

  Thus for all $t \in (0,T)$, we have

  $$Z(t) = e^{i\sqrt{\lambda}t}\begin{pmatrix} \frac{\varphi}{i\sqrt{\lambda}} \\ \varphi \end{pmatrix}.$$

  By assumption $(0, C^*)Z(t) = e^{i\sqrt{\lambda}t}C^*\varphi = 0$. From the definition of $\mathcal{N}_{\mathcal{Q}^*}(T)$, $\varphi \in \mathcal{N}_{\mathcal{Q}^*}(T) = \{0\}$, whence the contradiction.
- For every φ eigenvector of $\mathcal{Q}^* + aI_{\mathcal{H}}$, $C^*\varphi \neq 0$.

  Indeed, the couple (φ, λ) is a vector-eigenfunction of $\mathcal{Q}^*$ if and only if the couple $(\varphi, \lambda + a)$ is a vector-eigenfunction of $\mathcal{Q}^* + aI_{\mathcal{H}}$, and we can use the previous point.

Combining these 3 arguments provides Lemma 22.

■

Proof of Lemma 21. Since $\mathbb{K} = \mathbb{C}$, A^* is triangularisable. Hence, There exists an invertible matrix P such that $A^* = P\mathcal{T}P^{-1}$ with $\mathcal{T} = (t_{ij})$ some lower triangular matrix. Using the change of variables $V = P^{-1}Z$, we deduce that $\mathcal{N}_{Q_n^*+A^*}(T) = \{0\}$ if and only if $\mathcal{N}_{Q_n^*+\mathcal{T}^*}(T) = \{0\}$. The system

$$\begin{cases} Z_{tt} = Q_n^* Z + \mathcal{T}Z, \\ Z(0) = Z_0, \end{cases} \tag{5.44}$$

can be written as

$$\begin{cases} Z^1_{tt} = \mathcal{Q}^*_n Z^1 + t_{11} Z^1, \\ Z^2_{tt} = \mathcal{Q}^*_n Z^2 + t_{21} Z^1 + t_{22} Z^2, \\ \vdots \\ Z^n_{tt} = \mathcal{Q}^*_n Z^n + t_{n1} Z^1 + t_{n2} Z^2 + \cdots + t_{nn} Z^n, \\ Z(0) = Z_0. \end{cases} \tag{5.45}$$

Let $Z_0 = \begin{pmatrix} Z^1_0 \\ \vdots \\ Z^n_0 \end{pmatrix} \in \mathcal{N}_{Q^*_n + \mathcal{T}^*}(T)$. By definition $\overline{C}^*_n Z = \begin{pmatrix} \overline{\mathcal{C}}^* Z^1 \\ \vdots \\ \overline{\mathcal{C}}^* Z^n \end{pmatrix} = 0$ with Z solution of (5.44), hence $Z^1_0 \in \mathcal{N}_{\mathcal{Q}^* + t_{11} I_{\mathcal{H}}}$. Since $\mathcal{N}_{\mathcal{Q}^*}(T) = 0$ and using Lemma 22, we have $Z^1_0 = 0$. Thus the system can be written as

$$\begin{cases} Z^2_{tt} = \mathcal{Q}^*_n Z^2 + t_{22} Z^2, \\ \vdots \\ Z^n_{tt} = \mathcal{Q}^*_n Z^n + t_{n2} Z^2 + \cdots + t_{nn} Z^n, \\ Z(0) = Z_0. \end{cases}$$

Using the same reasoning as before by replacing Z^1 by $Z^2, Z^3, \ldots, Z^n$ successively, we get by induction that $Z^2_0 = \cdots = Z^n_0 = 0$, thus we have $Z_0 = 0$ and we infer that $\mathcal{N}_{Q^*_n + \mathcal{T}^*}(T) = 0$, which concludes the proof thanks to Lemma 20.

■

Algebraic problem :

For $f := \overline{C}_n(U)$, we want find a pair $(X, \tilde{W}) \in C^0([0,T]; \mathcal{D}(Q_n)) \cap C^1([0,T]; \mathcal{D}(Q_n^{\frac{1}{2}})) \cap C^2([0,T]; \mathcal{H}^n) \times C^0([0,T]; \mathcal{U}^m)$ satisfying the following control problem :

$$\begin{cases} \partial_{tt} X = Q_n(X) + AX + BC_m \tilde{W} + f & \text{in} \quad (0,T) \times \mathcal{H}^n, \\ X(0) = X(T) = \partial_t X(0) = \partial_t X(T) = 0. \end{cases} \tag{5.46}$$

As in the proof of Theorem 6, we will solve instead

$$\begin{cases} \partial_{tt} X = Q_n(X) + AX + BW + f & \text{in} \quad (0,T) \times \mathcal{H}^n, \\ X(0) = X(T) = \partial_t X(0) = \partial_t X(T) = 0, \end{cases} \tag{5.47}$$

with $W \in \mathcal{C}_m(\mathcal{U}^m)$. We will mimic the proof of Proposition 11. In the same way we can write Problem (5.34) under the abstract form

$$\tilde{\mathcal{P}}(X, W) = f, \tag{5.48}$$

where

$$\tilde{\mathcal{P}} : \begin{array}{lcc} \mathcal{D}(\tilde{\mathcal{P}}) \subset L^2(0,T; \mathcal{H}^{n+m}) & \to & L^2(0,T; \mathcal{H}^n) \\ (X\,,\, W) & \mapsto & \partial_{tt} X - Q_n X - AX - BW. \end{array} \tag{5.49}$$

The goal will be then to find a partial differential operator $\mathcal{M}$ satisfying

$$\tilde{\mathcal{P}} \circ \mathcal{M} = I_n. \tag{5.50}$$

Proposition 13. *Let $(A, B) \in \mathcal{M}_n(\mathbb{K}) \times \mathcal{M}_{n,m}(\mathbb{K})$. If rank$([A|B]) = n$, $\tilde{\mathcal{P}}$ has a right inverse* *denoted by $\mathcal{M}$. Moreover, the operator $\mathcal{M}$ is an operator of order*

$$\begin{cases} 2n-4 & \text{for } n \text{ first components} \\ 2n-2 & \text{for the } m \text{ last components} \end{cases} \tag{5.51}$$

in time and

$$\begin{cases} n-2 & \text{for } n \text{ first components} \\ n-1 & \text{for the } m \text{ last components} \end{cases} \tag{5.52}$$

in terms of powers of $\mathcal{Q}$.

Proof of Proposition 13. Changing ∂_t to ∂_{tt} in the proof of Proposition 11 we have

$$\tilde{\mathcal{P}} \circ \tilde{\mathcal{M}} = [A|B]$$

with

$$\begin{array}{rcl} \tilde{\mathcal{M}}: \ \mathcal{D}(\tilde{\mathcal{M}}) \subset L^2(0,T;\mathcal{H}^{nm}) & \to & L^2(0,T;\mathcal{H}^{n+m}) \\ f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \\ f_{m+1} \\ \vdots \\ f_{2m} \\ \vdots \\ f_{nm} \end{pmatrix} & \mapsto & \tilde{\mathcal{M}}f, \end{array}$$

defined by

$$\tilde{\mathcal{M}}f = \begin{pmatrix} \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} (-1)^{j+1}(-\partial_{tt} + Q_n)^j A^{i-1-j} B \begin{pmatrix} f_{jm+1} \\ \vdots \\ f_{(j+1)m} \end{pmatrix} \\ \\ \sum_{i=0}^{n-1} (-1)^{i+1}(-\partial_{tt} + Q_m)^i \begin{pmatrix} f_{im+1} \\ \vdots \\ f_{(i+1)m} \end{pmatrix} \end{pmatrix}. \tag{5.53}$$

Since $\mathrm{rank}([A|B]) = n$, there exists $D \in \mathcal{M}_{nm,n}$ such that $[A|B]D = I_n$. Introducing the operator $\mathcal{M} := \tilde{\mathcal{M}}D$ we obtain (5.50). Moreover, in the n first components the higher order term is $(-\partial_{tt} + \mathcal{Q})^{n-2}$ and in the m last components the higher order term is $(-\partial_{tt} + \mathcal{Q})^{n-1}$, which concludes the proof. ■

Conclusion : combination of the Analytic and Algebraic Problems.

The proof is similar to the one-order case, so that we just give the main arguments here. Let (X, W) be defined by

$$\begin{pmatrix} X \\ W \end{pmatrix} := \mathcal{M}\left(\overline{C}_n U\right), \tag{5.54}$$

with $\overline{C}_n U \in H_0^{2n-2}(0,T;\overline{C}_n(\mathcal{U}^n)) \bigcap \cap_{k=0}^{2n-2} C^k([0,T];\overline{C}_n\overline{C}_n^*(\mathcal{H}^n_{n-1-\frac{k}{2}}))$ constructed in Proposition 12. Using Proposition 13 and Assumption 5.1.2 , we obtain that

$$(X,W) \in \left(H^2_0(0,T;(\overline{C}_n(\mathcal{U}^n))) \bigcap \cap_{k=0}^2 C^k([0,T];\mathcal{H}^n_{1-\frac{k}{2}})\right) \times C^0([0,T],\overline{C}_m(\mathcal{U}^m)).$$

Notably, there exists $\tilde{W} \in C^0([0,T],\mathcal{U}^m)$ such that $W = C_m\tilde{W}$. Moreover, using Proposition 12 we have $X(0) = X(T) = 0$ in $\mathcal{H}^n$ and we remark that $(X,\tilde{W})$ is a solution to the problem

$$\begin{cases} \partial_{tt}X = Q_n(X) + AX + BC_m\tilde{W} + f & \text{in} \quad (0,T)\times\mathcal{H}^n, \\ X(0) = X(T) = \partial_t X(0) = \partial_t X(T) = 0. \end{cases}$$

Thus the pair $(Y,V) := (Z - X, -\tilde{W})$ is a solution to System (Ord2) in $C^0([0,T];\mathcal{D}(Q_n)) \cap C^1([0,T];\mathcal{D}(Q_n^{\frac{1}{2}})) \cap C^2([0,T];\mathcal{H}^n) \times C^0([0,T];\mathcal{U}^m)$ and satisfies

$$\begin{array}{c} Y(0) = Y^0 \ \text{ in } \ \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n, \quad \partial_t Y(0) = Y^1 \ \text{ in } \ \mathcal{D}(\mathcal{Q}^{n-1})^n \\ Y(T) \equiv 0 \ \text{ in } \ \mathcal{H}^n, \quad \partial_t Y(T) \equiv 0 \ \text{ in } \ \mathcal{H}^n. \end{array}$$

■

5.3.2 Second part of the proof of Theorem 7

In this section, we assume that the Kalman condition is NOT satisfied and we want to prove that the null controllability of (Ord2) fails. We will use an argument based on the *transmutation method* in order to go back to a parabolic one-order system. The ideas are then essentially the same as in Section 5.2.2.
From [79, Proposition 5.2] and using the transmutation technique (as in [79, Section 5.3]) we have the following lemma :

Lemma 23. *Let $S^* > 0$. If the system*

$$\left\{ \partial_{ss}Y = Q_nY + AY + BC_mU \qquad \textit{in} \quad \mathbb{R}_+ \times \mathcal{H}^n, \right. \tag{5.55}$$

is null controllable for every $S > S^*$ then at any time $T > 0$, for every initial condition $Z^0 \in \mathcal{H}^n$, there exists a control $V \in C^0([0,T];\mathcal{U}^m)$ such that the solution Z to System

$$\begin{cases} \partial_t Z = Q_nZ + AZ + BC_mV & \textit{in} \quad (0,T)\times\mathcal{H}^n, \\ Z(0) = Z^0, \end{cases} \tag{5.56}$$

satisfies

$$Z(T) \equiv 0 \ \text{ in } \ \mathcal{H}^n.$$

Proof of Lemma 23. Let $\epsilon > 0$ and $T \in]0, \inf(1,(S^*)^2)[$. From [79, Proposition 5.2], we can construct a function $v \in L^2(\mathbb{R}^2)$ such that

$$\partial_t v - \partial_{ss} v = 0 \quad \text{in} \quad \mathcal{D}'(]0,\infty[\times]-S,S[), \tag{5.57}$$

$$v_{|t=0} = \delta_0 \quad \text{and} \quad v_{|t=T-\epsilon T} = 0, \tag{5.58}$$

$$v(t,s) = 0 \quad \text{for every} \quad (t,s) \in (\mathbb{R}\backslash]0,T-\epsilon T[) \times (\mathbb{R}\backslash]-S,S[)\,.$$

Let $Z^0 \in \mathcal{H}^n$. We have $\tilde{Z}^0 = e^{\epsilon T(Q_n+A)}Z^0 \in \mathcal{D}(Q_n^{n-\frac{1}{2}})$. Let (Y,U) be a solution to System (5.55) in $C^0(\mathbb{R}_+;\mathcal{H}^n) \times C^0(\mathbb{R}_+;\mathcal{U}^m)$ with $(Y(0),\partial_sY(0)) = (\tilde{Z}^0, 0)$ such that $Y(S) = \partial_sY(S) = 0$ and

$Y(s) = 0$ for every $s \in [S, +\infty[$. We define $\underline{Y}$ and $\underline{U}$ as the extension of Y and U by reflection with respect to s, that is to say $\underline{Y}(s) = Y(s) = \underline{Y}(-s)$ and $\underline{U}(s) = U(s) = \underline{U}(-s)$ for every $s \in \mathbb{R}_+$. Thus, the pair $(\underline{Y}, \underline{U})$ satisfies

$$\partial_{ss}\underline{Y} = Q_n\underline{Y} + A\underline{Y} + BC_m\underline{U} \quad \text{in} \quad \mathbb{R} \times \mathcal{H}^n. \tag{5.59}$$

Let $Z \in L^2(\mathbb{R}, \mathcal{D}(Q_n))$ and $V \in L^2(\mathbb{R}, \mathcal{U}^m)$ defined by

$$Z(t) = \int_{\mathbb{R}} v(t - \epsilon T, s)\underline{Y}(s)ds \quad \text{and} \quad V(t) = \int_{\mathbb{R}} v(t - \epsilon T, s)\underline{U}(s)ds.$$

Since $\underline{Y}(S) = \partial_s\underline{Y}(S) = 0$, the equations (5.57) and (5.59) imply

$$\partial_t Z = Q_n Z + AZ + BC_m V \quad \text{in} \quad]\epsilon T, \infty[\times \mathcal{H}^n.$$

The conditions (5.58) imply then

$$Z_{|t=\epsilon T} = \tilde{Z}^0 \quad \text{and} \quad Z_{|t=T} = 0,$$

whence the conclusion. ■

We assume that $\operatorname{rank}([A|B]) \neq n$. Thus, there exists $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \in \mathcal{H}^n \backslash \{0\}$ such that $B^*X = B^*A^*X = \cdots B^*(A^*)^{n-1} X = 0$. We consider the following adjoint system of (5.56)

$$\begin{cases} -\partial_t\varphi = Q_n^*(\varphi) + A^*\varphi & \text{in} \quad (0,T) \times \mathcal{H}^n, \\ \varphi(T) = \varphi^T. \end{cases} \tag{5.60}$$

It is well-known that System (5.56) is null controllable at time T if and only if there exists a positive constant C_1 such that for all solution φ of (5.60)

$$\|\varphi(0)\|_{\mathcal{H}^n} \leqslant C_1 \int_0^T \|C_m^* B^*\varphi\|_{\mathcal{U}^m}\, dt, \tag{5.61}$$

for all $\varphi^T \in \mathcal{D}(L_n^*)$. Let φ_X the solution of (5.60) with $\varphi_X(T) = X$. Let $S = (S_t)_{t\in\mathbb{R}}$ be a strongly continuous semigroup on $\mathcal{H}$, with generator $\mathcal{Q} : \mathcal{D}(\mathcal{Q}) \subset \mathcal{H} \to \mathcal{H}$. By the definition of Q_n given in (5.6), we obtain $C_m^*B^*\varphi_X = C_m^*B^*e^{tA^*}\begin{pmatrix} S^*(T-t)X_1 \\ \vdots \\ S^*(T-t)X_n \end{pmatrix}$. Since $(A, B) \in \mathcal{M}_n(\mathbb{K}) \times \mathcal{M}_{n,m}(\mathbb{K})$, for all $i \in \{0, \cdots, n-1\}$, $B^*(A^*)^i \begin{pmatrix} S^*(T-t)X_1 \\ \vdots \\ S^*(T-t)X_n \end{pmatrix}$ is solution of

$$\begin{cases} -\partial_t\varphi = Q_m^*(\varphi) & \text{in} \quad (0,T) \times \mathcal{H}^m, \\ \varphi(T) = B^*(A^*)^i X = 0. \end{cases}$$

Thus

$$B^*(A^*)^i \begin{pmatrix} S^*(T-t)X_1 \\ \vdots \\ S^*(T-t)X_n \end{pmatrix} = 0.$$

Using the Cayley-Hamilton theorem, we obtain that $C_m^* B^* \varphi_X = 0$, from which we deduce that (5.61) is not satisfied. Thus, the System (5.56) is not controllable at time T. Using Lemma 23, we deduce that the system

$$\partial_{tt} Y = Q_n Y + AY + BC_m U \qquad \text{in} \quad (0,T) \times \mathcal{H}^n,$$

is not controllable, which concludes the proof. ■

5.4 Applications

In this section, we will give some examples of applications in the case of systems of Schrödinger and wave equations with internal control. Let Ω be a smooth bounded open subset of $\mathbb{R}^N$. We will denote by $L^2(\Omega)$ the set of square-integrable functions defined on Ω with values in the complex plane $\mathbb{C}$.

We recall the following definition that will be widely used in what follows.

Definition 1 (GCC)**.** *We say that ω verifies the Geometric Optics Condition (GCC in short) if there exists $T^* > 0$ such that any rays of Geometric Optics in Ω enter the open set ω in time smaller than T^* (see [15]) .*

5.4.1 System of Schrödinger equations with internal control

Let us introduce the state space $\mathcal{H} = L^2(\Omega)$ and the control space $\mathcal{U} = L^2(\Omega) = \mathcal{H}$. We consider the linear continuous operator $\mathcal{C} : L^2(\Omega) \to L^2(\Omega)$ defined by $\mathcal{C}u = \mathbb{1}_\omega u$ with

$$\mathbb{1}_\omega = \begin{cases} 1 & \text{on} \quad \omega, \\ 0 & \text{on} \quad \Omega \backslash \omega. \end{cases}$$

We consider the following Schrödinger equation

$$\begin{cases} \partial_t z = i\Delta z + W(x)z + \mathbb{1}_\omega u, \\ z(0) = z_0, \end{cases} \tag{5.62}$$

where the potential W is in $C^\infty(\bar{\Omega}; \mathbb{K})$. It is well-known that the operator $i\Delta + W(x) : \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is a generator of a group on $L^2(\Omega)$. Let $n \in \mathbb{N}^*$, we have $\mathcal{D}((i\Delta + W(x))^n) = \mathcal{D}(((i\Delta + W(x))^*)^n) = H_{(0)}^{2n}(\Omega)$ where $H_{(0)}^{2n}(\Omega)$ is defined by

$$H_{(0)}^{2n}(\Omega) := \{v \in H^{2n}(\Omega) \text{ such that } v = \Delta v = \cdots = \Delta^{n-1} v = 0 \text{ on } \partial\Omega\}. \tag{5.63}$$

We consider the control system

$$\begin{cases} \partial_t Y = i\Delta Y + W(x)Y + AY + \mathbb{1}_\omega BV & \text{in} \quad (0,T) \times L^2(\Omega)^n, \\ Y(0) = Y^0, \end{cases} \tag{5.64}$$

with $(A, B) \in \mathcal{M}_n(\mathbb{K}) \times \mathcal{M}_{n,m}(\mathbb{C})$.

Our first result concerns the case where ω is strongly included in Ω.

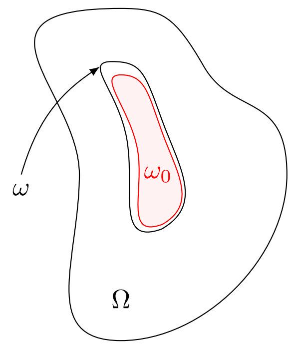

FIGURE 5.1 – An example when $\overline{\omega} \subset \Omega$.

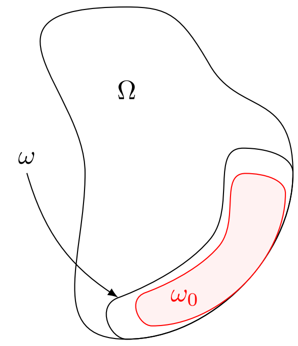

FIGURE 5.2 – An example when $\omega \subset \overline{\Omega}$ is an open set such that $\omega \cap \partial\Omega \neq \emptyset$.

Theorem 8. *Let $T > 0$ and let us assume that the open set ω of Ω satisfy GCC and $\overline{\omega} \subset \Omega$, then there exists a control V in $C^0((0,T) \times (L^2(\Omega))^m)$ such that the solution of (5.64) with initial condition $Y(0) = Y^0$ in $(H^{2n-2}_{(0)}(\Omega))^n$ satisfies*

$$Y(T) \equiv 0 \quad in \quad (L^2(\Omega))^n$$

if and only $Y^0 \in [A|B]((L^2(\Omega))^{nm})$.

Proof of Theorem 8. Since $\overline{\omega} \subset \Omega$, one can construct a function $\tilde{\mathbb{1}}_\omega \in \mathcal{C}^\infty(\Omega)$ defined by

$$\tilde{\mathbb{1}}_\omega := \begin{cases} 1 & \text{on} \quad \omega_0, \\ 0 & \text{on} \quad \Omega \backslash \omega, \end{cases} \tag{5.65}$$

where ω_0 is some well-chosen open set strongly included in ω still verifying GCC, so that

$$\begin{cases} \partial_t z = (i\Delta + W(x))z + \tilde{\mathbb{1}}_\omega u, \\ z(0) = z_0, \end{cases} \tag{5.66}$$

is exactly controllable thanks to the result of [66] (see Figure 5.1). Thus, we deduce that Assumption 5.1.1 is verified with $\overline{\mathcal{C}} : L^2(\Omega) \to L^2(\Omega)$ defined by $\overline{\mathcal{C}}u = \tilde{\mathbb{1}}_\omega u$.
To apply Theorem 6 we need to verify Assumption 5.1.2 where $\overline{\mathcal{C}}$ is described above. Let $\varphi \in \mathcal{D}(((i\Delta + W(x))^*)^k) = H^{2k}_{(0)}(\Omega)$, for all $k \in \{0, \cdots, n-1\}$, by definition of $\tilde{\mathbb{1}}_\omega \in \mathcal{C}^\infty_c(\Omega)$ we obtain $\overline{\mathcal{C}}\overline{\mathcal{C}}^*\varphi = \tilde{\mathbb{1}}^2_\omega \varphi \in H^{2k}_{(0)}(\Omega)$ and $(i\Delta + W(x))^k(\overline{\mathcal{C}}\overline{\mathcal{C}}^* \mathcal{D}(((i\Delta + W(x))^*)^k) \in C(L^2(\Omega))$. Thus, we can apply Theorem 6 whence the conclusion of the proof of Theorem 8. ■

Remark 5. One can obtain the same result as in Theorem 8 with exactly the same proof by replacing the open set Ω with some regular compact connected Riemannian manifold without boundary M, with ω any open subset of M verifying GCC.

From now on we consider ω an open subset of $\overline{\Omega}$ such that $\omega \cap \partial\Omega \neq \emptyset$ (see Figure 5.2). In general it is impossible to construct a function $\tilde{\mathbb{1}}_\omega \in \mathcal{C}^\infty(\Omega)$ as (5.65) such that the system (5.66) is controllable and $\tilde{\mathbb{1}}_\omega$ maps $H^{2k}_{(0)}(\Omega)$ into itself for all $k \in [3, \infty]$. More precisely, satisfying Assumption 5.1.2 would require that, for every $\varphi \in H^{2(n-1)}_{(0)}(\Omega)$, for every $k \leqslant n-2$, $\Delta^k(\tilde{\mathbb{1}}_\omega \varphi) = 0$ on $\partial\Omega$ which is not verified without additional conditions on $\tilde{\mathbb{1}}_\omega$. If $n = 2$ it is clear that $\tilde{\mathbb{1}}_\omega$ maps $L^2(\Omega, \mathbb{C})$ into itself. If $n = 3$, we need to assume moreover that $\nabla \tilde{\mathbb{1}}_\omega . \vec{n} = 0$ with $\vec{n}$ the unit normal vector. If $n > 3$, $\tilde{\mathbb{1}}_\omega$ has to satisfy strong global geometric conditions, for instance $\tilde{\mathbb{1}}_\omega$ can be chosen constant near any connected component of the boundary (for more informations we refer to [29, Section 4.2]). However, we cannot affirm anymore that the system (5.66) will still be controllable without loss of regularity or locality on the function $\tilde{\mathbb{1}}_\omega$, so that Assumption 5.1.2 will not be verified. Hence, we will focus our attention on some particular cases. We will first consider the case where the number of equations is less than or equal to three without additional conditions on Ω and then we will consider the case where Ω is the product of N open intervals in $\mathbb{R}^N$ and the case where Ω is a unit disk.

Theorem 9. *Let $T > 0$. Let ω be an open subset of $\overline{\Omega}$ such that $\omega \cap \partial\Omega \neq \emptyset$ and ω satisfies GCC. Let $n \leqslant 3$, then there exists a control V in $C^0([0,T];(L^2(\Omega))^m)$ such that the solution of (5.64) corresponding to the initial condition $Y(0) = Y^0$ in $(H^{2n-2}_{(0)}(\Omega))^n$ satisfies*

$$Y(T) \equiv 0 \quad \textit{in} \quad (L^2(\Omega))^n,$$

if and only $Y^0 \in [A|B]((L^2(\Omega))^{nm})$.

Proof of Theorem 9. We can keep constructing a function $\tilde{\mathbb{1}}_\omega \in \mathcal{C}^\infty(\Omega)$ defined by

$$\tilde{\mathbb{1}}_\omega := \begin{cases} 1 & \text{on} \quad \omega_0, \\ 0 & \text{on} \quad \Omega\backslash\omega, \end{cases} \tag{5.67}$$

where ω_0 is some well-chosen open set included in ω still verifying GCC such that $\nabla\tilde{\mathbb{1}}_\omega.\vec{n} = 0$ and the system

$$\begin{cases} \partial_t z = i\Delta z + W(x)z + \tilde{\mathbb{1}}_\omega u, \\ z(0) = z_0, \end{cases} \tag{5.68}$$

is exactly controllable (see Figure 5.2). Moreover, using [29, Section 4.2] and $\nabla\tilde{\mathbb{1}}_\omega.\vec{n} = 0$, we infer that $\mathbb{1}^2_\omega$ maps $H^{2k}_{(0)}(\Omega)$ into itself for $k \leqslant 2$ and by definition of $\tilde{\mathbb{1}}_\omega$, we have $(i\Delta + W(x))^k(\overline{\mathcal{C}\mathcal{C}}^*\mathcal{D}(((i\Delta + W(x))^*)^k)) \in C(L^2(\Omega))$ for $k \leqslant 2$. Thus, Assumption 5.1.2 is verified and we can conclude as in the proof of Theorem 8. ∎

It is well-known that GCC is only a sufficient condition to ensure the exact controllability of the Schrödinger equation, but is not always necessary in some particular geometries. Let us give two examples. If Ω is the product of N open intervals, we do not need to impose restrictions on ω and if Ω is a unit disk, ω has to touch the boundary of Ω.

Theorem 10. *Let $T > 0$ and let us assume that the domain $\Omega \subset \mathbb{R}^N$ is the product of N open intervals, then there exists a control V in $C^0((0,T) \times (L^2(\Omega))^m)$ such that the solution of (5.64) corresponding to the initial condition $Y(0) = Y^0$ in $(H^{2n-2}_{(0)}(\Omega))^n$ satisfies*

$$Y(T) \equiv 0 \quad \textit{in} \quad (L^2(\Omega))^n,$$

if and only $Y^0 \in [A|B]((L^2(\Omega))^{nm})$.

Proof of Theorem 10. From [62, Proposition 8.8], one can easily deduce that the system

$$\begin{cases} \partial_t z = i\Delta z + W(x)z + \mathbb{1}_{\tilde{\omega}} u, \\ z(0) = z_0, \end{cases}$$

is exactly controllable for any nonempty open $\tilde{\omega}$ subset of Ω. Consequently, without loss of generality we can assume that $\overline{\omega} \subset \Omega$. Thus, we just have to copy the proof of Theorem 8 and we deduce the expected result. ∎

Theorem 11. *Let $T > 0$ and let us assume that the domain $\Omega \subset \mathbb{R}^2$ is the unit disk and let ω be an open subset of $\overline{\Omega}$ such that $\omega \cap \partial\Omega \neq \emptyset$. Let $n \leqslant 3$, then there exists a control V in $C^0([0,T];(L^2(\Omega))^m)$ such that the solution of (5.64) corresponding to the initial condition $Y(0) = Y^0$ in $(H^{2n-2}_{(0)}(\Omega))^n$ satisfies*

$$Y(T) \equiv 0 \quad in \quad (L^2(\Omega))^n,$$

if and only $Y^0 \in [A|B]((L^2(\Omega))^{nm})$.

Proof of Theorem 11. Since the domain $\Omega \subset \mathbb{R}^2$ is the unit disk, from [12, Theorem 1.2], we deduce that the equation

$$\begin{cases} \partial_t z = i\Delta z + W(x)z + \mathbb{1}_{\tilde{\omega}} u, \\ z(0) = z_0, \end{cases}$$

is exactly controllable for any open $\tilde{\omega}$ subset of $\overline{\Omega}$ such that $\tilde{\omega} \cap \partial\Omega \neq \emptyset$. Thus, we just mimic the proof of Theorem 9 and we have directly the expected result. ∎

Remark 6. The same results can be obtained by replacing the Schrödinger equation by the plate equation (see for example [66, Section 5]).

5.4.2 System of wave equations with internal control

We consider the state space and control space as in the previous section, i.e. $\mathcal{H} = \mathcal{U} = L^2(\Omega)$. We introduce the operator $\mathcal{C} : L^2(\Omega) \to L^2(\Omega)$ such that $\mathcal{C}u = \mathbb{1}_\omega u$. We consider the following wave equation

$$\begin{cases} \partial_{tt} z = \Delta z + W(x)z + \mathbb{1}_\omega u, \\ z(0) = z_0, \\ \partial_t z(0) = z_1, \end{cases} \tag{5.69}$$

where the potential W is in $C^\infty(\bar{\Omega};\mathbb{K})$. It is well-known that the operator $\Delta + W(x) : \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is self-adjoint with compact resolvent, but it is not a negative operator in general. This is not a problem here because we know that for $\mu > 0$ large enough the operator $\Delta + W(x) - \mu$ becomes negative, hence we can adapt the results of Theorem 7 in this case. Moreover, we know that if ω verifies GCC, then (5.69) is controllable at any time $T > T^*$, where T^* is the minimal time needed to ensure that all the rays of Geometric Optics in Ω enter the open set ω. Let $n \in \mathbb{N}^*$, we have $\mathcal{D}((\Delta + W(x))^n) = H^{2n}_{(0)}(\Omega)$ where $H^{2n}_{(0)}(\Omega)$ is defined in (5.63). We consider the control system

$$\begin{cases} \partial_{tt} Y = \Delta Y + W(x)Y + AY + \mathbb{1}_\omega BV \quad & \text{in} \quad (0,T) \times L^2(\Omega)^n, \\ Y(0) = Y^0, \\ \partial_t Y(0) = Y^1, \end{cases} \tag{5.70}$$

with $(A,B) \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_{n,m}(\mathbb{C})$. Mimicking the proof of Theorem 8 and Theorem 9 we immediately obtain the following results :

Theorem 12. *Let us assume that ω satisfies GCC and $\overline{\omega} \subset \Omega$. Let $(Y^0, Y^1) \in H^{2n-1}_{(0)}(\Omega) \times H^{2n-2}_{(0)}(\Omega)$. Then, for every $T > T^*$, the control system (5.70) is exactly controllable at time T if and only if rank$([A|B]) = n$.*

Remark 7. As in the case of the Schrödinger equation, one can obtain the same result as in Theorem 12 with exactly the same proof by replacing the open set Ω with some regular compact connected Riemannian manifold without boundary M, with ω any open subset of M verifying GCC.

Theorem 13. *Let ω be an open subset of $\overline{\Omega}$ such that $\omega \cap \partial\Omega \neq \emptyset$ and ω satisfies GCC. Let $n \leqslant 3$. Let $(Y^0, Y^1) \in H^{2n-1}_{(0)}(\Omega) \times H^{2n-2}_{(0)}(\Omega)$, then, for every $T > T^*$, the control system* (5.70) *is exactly controllable at time T if and only if rank$([A|B]) = n$.*

Acknowledgements

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Conclusion

Nous avons obtenu des estimations de la constante d'observabilité $c(Ta, \omega)$ (par dualité, du coût du contrôle) en utilisant une méthode par propagation et une méthode spectrale. La version spectrale de la constante d'observabilité nous a amené à déterminer une borne inférieure du problème d'optimisation

$$\inf_{0 \leqslant a \leqslant M} \inf_{|\omega| = rL} \inf_{j \in \mathbb{N}^*} \int_\omega e_{a,j}(x)^2 \, dx,$$

avec $e_{a,j}$ le j-ième vecteur propre de l'opérateur $-\Delta + a(\cdot)$. Nous avons vu que cette estimation est optimale par rapport à la mesure de ω. On montre, par exemple, que

$$\frac{c(T, a, \omega)}{|\omega|^3} = O(1), \quad \text{lorsque} \quad |\omega| \mapsto 0.$$

Dans un deuxième temps, nous avons montré des résultats de contrôlabilité indirecte pour certains systèmes de n équations et de m contrôles avec $m < n$. Le contrôle agit indirectement sur les $n-m$ équations au travers d'une matrice constante A. Pour le système d'équations de Schrödinger, on a caractérisé les données initiales qui sont contrôlâbles et pour le système d'équations des ondes on montre la contrôlabiltité à 0 sous des conditions nécessaires et suffisantes de type Kalman.

Une question intéressante serait d'obtenir une estimation du coût du contrôle pour des systèmes d'équations en utilisant une décomposition spectrale de la constante d'observabilité. Par exemple, supposons que ω est un ouvert inclus dans $(0, L)$ vérifiant GCC et considérons un système à n équations des ondes et à m contrôles de la forme

$$\partial_{tt} Y - \Delta Y = AY + \chi_\omega BU(t, x), \quad (t, x) \in (0, T) \times (0, L), \tag{Syst-ondes}$$

avec $\chi_\omega : (0, L) \to \mathbb{R}$ une fonctions régulière sur $(0, L)$ à support dans ω, $B \in \mathcal{M}_{mn}(\mathbb{R})$ et $A \in \mathcal{M}_n(\mathbb{R})$ est une matrice symétrique. Si $\mathrm{rang}(B, AB, \cdots, A^{n-1}B) = n$, sous des conditions sur ω, d'après [Chapitre 5, Théorème 12 et Théorème 13], le système (Syst-ondes) est contrôlable. De plus, il existe une base hilbertienne de vecteurs propres $(e_{A,j})_{j \in \mathbb{N}^*}$ associés à l'opérateur $-\Delta + A$ sur $L^2(\Omega)^n$. Pouvons-nous déterminer une estimation fine du coût du contrôle pour (Syst-ondes) en utilisant la décomposition spectrale de la solution Y ?

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Observation et contrôle de quelques systèmes conservatifs

Résumé

Dans cette thèse, nous nous intéressons à la contrôlabilité interne et à son coût pour une ou plusieurs équations aux dérivées partielles conservatives.
Dans la première partie, nous introduisons et détaillons deux méthodes permettant d'estimer le coût du contrôle (et par dualité, de la constante d'observabilité) de l'équation des ondes avec potentiel L^∞ en dimension un d'espace. La première utilise la propagation des ondes le long des caractéristiques en s'appuyant sur le rôle symétrique de la variable de temps et d'espace. La deuxième méthode repose sur la décomposition spectrale de l'équation des ondes et sur l'utilisation des inégalités d'Ingham. L'estimation de la constante d'observabilité se ramène alors à l'étude d'un problème d'optimisation faisant intervenir les vecteurs propres du Laplacien-Dirichlet avec potentiel. Nous fournissons ensuite des propriétés qualitatives sur le minimiseurs ainsi qu'une estimation du minimum ne dépendant que de la mesure de l'ensemble d'observation.
Dans la deuxième partie, nous étudions la contrôlabilité de certains systèmes d'équations avec un nombre de contrôles réduits, autrement dit le nombre de contrôles est plus petit que le nombre d'équations. En particulier, nous caractérisons exactement les données initiales qui peuvent être contrôlées pour des systèmes d'équations couplées de type Schrödinger et nous énonçons une condition nécessaire et suffisante de type Kalman pour des systèmes d'équations des ondes couplées. La preuve repose sur une méthode de contrôle fictif combinée à la résolution algébrique d'un système sous-déterminé et sur certains résultats de régularité.

Mots clés : opérateurs de sturm-liouville, problèmes extrémaux, calculs des variations, inégalité d'ingham, contrôlabilité indirecte de systèmes linéaires, équations des ondes et de schrödinger, méthode de contrôle fictif, résolubilité algébrique.

Abstract

In this work, we focus on the internal controllability and its cost for some linear partial differential equations.
In the first part, we introduce and describe two methods to provide precise estimates of the cost of control (and by duality, of the observability constant) for general one dimensional wave equations with potential. The first one is based on a propagation argument along the characteristics relying on the symmetrical roles of the time and space variables. The second one uses a spectral decomposition of the solution of the wave equation and Ingham's inequalities. This relates the estimation of the observability constant to the study of an optimal problem involving Dirichlet eigenfunctions of Laplacian with potential. We provide some qualitative properties of the minimizers, and also precise bounds on the minimum.
In the second part, we are concerned with the controllability of some systems of equations by a reduced number of controls (i.e. the number of controls is less that the number of equations). In particular, in the case of coupled systems of Schrödinger equations, we exactly characterize the initial conditions that can be controlled and we give a necessary and sufficient condition of Kalman type for the controllability of coupled systems of wave equations. The proof relies on the *fictitious control method* coupled with the proof of an *algebraic solvability* property for some related underdetermined system, as well as on some regularity results.

Keywords: sturm-liouville operators, extremal problems, calculus of variations, ingham's inequality, indirect controllability of linear systems, schrödinger and wave equations, fictitious control method, algebraic solvability.

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